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On Relativistically Rigid Surfaces of Revolution

BY

J. R. POUNDER

INSTITIÚID ÁRD-LÉINN BHAILE ÁTHA CLIATH
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Summary: A definition of superficial rigidity recently proposed by Synge is applied to what appears to some Galileian observer as a surface of revolution with its axis of symmetry fixed. It is found that the metrical distortions of the meridians and the parallels of latitude of such a moving rigid surface relative to the corresponding surface at rest in a Galileian reference system are governed by relations analogous to the FitzGerald-Lorentz contraction rule. The special cases of (1) uniform rotation about the axis and (2) uniform screw motion along the axis are treated in detail. In (1) there is a radial contraction without change of meridian arc-length; in (2) there is in addition a uniform axial contraction and twist about the axis. If the axial component of velocity in (2) is made to approach the speed of light, the angular velocity remaining finite, then the moving surface shrinks both radially and axially, ultimately having the shape of an infinitesimal circular cylinder with flat ends; at the same time the length of the twisted curves corresponding to the original meridians remains finite. These conclusions are verified by taking the initial surface at rest to be a sphere and a cylinder; the meridian sections of the corresponding surfaces in motion are computed and shown in graphs. When applied to certain surfaces in uniform rotation, in particular multiply-connected ones, the conditions of superficial rigidity are mutually contradictory, unless one allows the formation of <u>edges</u>.

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L I S T O F F I G U R E S

LIST OF SYMBOLS

σ	In Sections 1 and 2, σ denotes any moving rigid surface; in Section 3 and Appendix A, a rigid surface moving symmetrically about a fixed axis of symmetry; elsewhere, a rigid surface of revolution in uniform rotation
σ_0	surface at rest corresponding to σ
a_1, a_2	Lagrangian labels for particles A of σ and σ_0 (e.g. s, θ_0 in Sections 3, 4, 5)
s	arc-length along meridian of σ_0
x_1, x_2, x_3	Cartesian coordinates on σ in frame of reference S
$\overset{o}{x}_1$, $\overset{o}{x}_2$, $\overset{o}{x}_3$	Cartesian coordinates on σ_0 in frame of reference S_0
$x_4 = ict$	t, time relative to frame S ; c, speed of light
$u_\rho = \dfrac{\partial x_\rho}{\partial t}$	velocity of particle with coordinates x_ρ ; $u^2 = \sum_1^3 u_\rho u_\rho$
v_t, v_n	tangential and normal components of u_ρ (relative to σ)
r, θ, z	cylindrical polar coordinates on σ
r_0, θ_0, z_0	cylindrical polar coordinates on σ_0
ω	angular velocity of σ about axis (constant from Section 4 on)
σ'	surface of revolution, corresponding to σ_0 and σ, in uniform screw motion
x'_ρ, r', θ', z'	Cartesian and polar coordinates on σ'
ω', U	angular velocity and axial velocity of screw motion of σ' ; $\gamma^2 = (1 - U^2/c^2)^{-1}$; $\omega' = \omega/\gamma$
P_0, P, P'	parallels of latitude on σ_0, σ, and σ'
M_0, M, M'	meridians on σ_0, σ, and σ'
N, N'	curves on σ, σ' corresponding to meridians M_0 on σ_0
Γ_0, Γ, Γ'	curves on σ_0, σ, σ' corresponding to one another and having same arc-length
m_0, m, m'	meridian <u>sections</u> of σ_0, σ, and σ'
α_0, α, α'	angles made by normals to m , m_0 , m' with axis of symmetry
$2\ell_0$	length of meridian section m_0
R_0	radius of sphere (Section 6) or cylinder (Section 7)
$\bar{r}_0$, $\bar{z}_0$, $\bar{r}$, $\bar{z}$, $\bar{r}'$, $\bar{z}'$, $\bar{s}$, $\bar{R}_0$	corresponding unbarred quantities multiplied by ω/c
$\bar{m}'$	meridian section m' magnified in the ratio ω/c : 1

ON RELATIVISTICALLY RIGID SURFACES OF REVOLUTION

1. Introduction

As a possible extension of the notion of rigidity to the special theory of relativity, Synge (1953) has recently defined a superficially rigid body. In this paper I shall consider some kinematical consequences of the definition, particularly for an axially symmetric body rotating uniformly round its axis.

A moving rigid body in Newtonian mechanics may be regarded as a continuum of particles in motion such that the Euclidean distance between any two particles is independent of the time. From the relativistic point of view this property is not absolute but relates to a particular frame of reference. The simplest definition of absolute rigidity is that of Born (1909), according to which two particles are rigidly connected if their world lines in space-time are equidistant, in the sense that their normal Minkowskian separation (i.e. the ordinary distance between the particles as measured in the instantaneous rest-frame of one of them) remains constant throughout the motion. A continuum of particles all rigidly connected in this way is then a rigid body in the sense of Born.

Any definition of rigid body based on the assigning of constant "distances" to pairs of particles can be extended in the following way. If between the particles of two rigid bodies with different motions there exists a one-to-one correspondence or mapping such that the "distances" between corresponding pairs of particles are equal, then the two bodies will be said to correspond, and the two motions may be regarded as two possible motions of a single rigid body, even though nothing may be known of the transition between them. In particular one of the two motions may be, in some Galileian frame of reference, a state of rest. Relative to a particular definition of "distance", three related questions now arise:

(1) How many degrees of freedom has a rigid body, i.e. how many arbitrary functions can occur in the equations describing its motion?

(2) What motions may a rigid body B have if it is to correspond to another body B_0 at rest?

(3) How does the instantaneous appearance of B differ from that of B_0 ?

(We need not distinguish between two bodies whose histories in space-time can be made to coincide by a transformation that leaves all "distances" invariant.)

The answers are immediate for the Newtonian rigid body: it has six degrees of freedom, its angular velocity and the motion of one of its particles being arbitrary functions of the time; all corresponding bodies are congruent, no matter what their motion. On the other hand the Born rigid body has in general only three degrees of freedom; but it may also move in such a way that no corresponding body could be at rest. In fact the world lines of the particles either (a) form a normal congruence in space-time, or (b) have constant curvatures and correspond to uniform screw motions in space (see Herglotz 1910). Even a finite number of discrete particles all rigidly connected to one another can be given only specially restricted motions - such a configuration is "over-rigid".

The possibility of restoring the additional freedom of motion by applying Born's definition of rigid connexion only to the surface of the body was suggested to Synge by the observation that in simple motions (see Ives 1945, Galli 1952) one could answer question (3) above for the surface of the body without considering the interior at all, and that in any case one can hardly justify the assumption of flat space-time in the interior of an accelerated body. One is thus led to the notion of a superficially rigid body, its surface being locally rigid in the sense that every pair of adjacent particles satisfies Born's rigidity condition; the problem is then to investigate the freedom of motion of such a rigid surface. For slow motions one obtains six degrees of freedom by taking the Newtonian rigid motion as a first approximation, and the general validity of this result is at least plausible: it should be possible to impart to any selected particle an arbitrary motion and move the adjacent particles with three more degrees of freedom by giving the Eulerian angles of a triad orthogonal to the world line of the first particle and requiring that nearby particles on two of the arms of the triad remain fixed to it.

The motions discussed in this paper are however mainly very special, and relate rather to the question of corresponding rigid surfaces: we seek the relation between two such surfaces σ_0 and σ, where σ_0 is at rest in a Galileian frame of reference S_0 and σ has a more or less prescribed motion relative to a frame of reference S . We may regard σ as the instantaneous appearance to the observer S of a moving surface that was formerly at rest (σ_0) , and speak of the distortion of σ relative to σ_0 (as in the FitzGerald-Lorentz contraction due to uniform translation). This will be, in the first place, a material distortion, i.e. a comparison of surface elements consisting of the same particles; each element in fact undergoes the FitzGerald-Lorentz contraction appropriate to the local velocity. When these relations are applied to surfaces of revolution σ_0 , σ (the motion of σ being symmetrical about its axis), which are characterized as to their form by meridian sections m_0 and m , they yield other analogous relations, e.g. between the elements of the respective meridian sections cut off by corresponding parallels of latitude; it turns out that m_0 and m are related exactly as they would be if σ had no angular velocity about its axis.

In order to obtain from these local distortions the deformation of a finite surface, the motion of σ is specialized to be, in particular: (i) a uniform rotation about the axis; (ii) a uniform screw motion along the axis, the results in (ii) being obtained from those in (i) by Lorentz transformation. In each case the form of σ is fixed in time. Certain conditions must be fulfilled by the form of σ if σ_0 is to exist, and by that of σ_0 if singular lines (or edges) are not to arise on σ. The effect of rotation on a surface of revolution is to produce a radial contraction without change of meridian arc-length; an axial component of velocity enhances this effect and superposes on it a uniform axial contraction and twist. An increase in speed of rotation leads to a needle-shaped figure in the limit, while an increase in the axial velocity brings about a simultaneous shrinking of all linear dimensions of σ , in such a way that its shape approaches that of a circular cylinder; owing to the accompanying increase in twist, however, there are corresponding material curves on σ_0 and σ whose lengths remain finite even in the limit.

These deformations are worked out in detail for spheres and cylinders (including disks), the results being presented graphically.

Some implications and possible extensions of the theory are briefly discussed in the concluding section.

2. The conditions for superficial rigidity

The only observations of moving particles we consider are those of special relativity: relative to any inertial frame of reference we can determine the positions of any number of particles at any instant and hence their mutual distances. The history of these particles in Minkowskian space-time is a congruence of world lines whose intersection by a hyperplane $t = t_0$ yields the instantaneous configuration in a particular reference frame S. The distance d between two particles in such a configuration will depend on the choice of S, and also in general on the instant t_0. If, in a fixed reference system S', d remains constant throughout the motion, then relative to S' the particles are rigidly connected in the Newtonian sense. They are rigidly connected in the relativistic sense if d remains constant when S is always taken to be the instantaneous rest-frame of one of the particles; in the rest-frame of the other particle d has then the same value. The world lines of the two particles thus have a single infinity of common normals (in the Minkowskian sense), all of length d, and are said to be equidistant, d being the distance between them.

In a relativistically rigid material continuum adjacent particles must be rigidly connected, their world lines being separated by constant infinitesimal distances; we shall now express this condition analytically. Relative to the reference frame S the coordinates x_i of the particles of a continuum C are continuously differentiable functions of a current parameter τ and of one or more Lagrangian labels. Here x_ρ are rectangular Cartesian coordinates and $x_4 = ict$, the Minkowskian metric being $dx_i\, dx_i = dx_\rho\, dx_\rho - c^2\, dt^2$. (Throughout the paper Latin indices take the values 1 to 4, Greek indices 1 to 3, with the summation convention for repeated indices.) The motion and resulting distortion being described most naturally in a definite frame, the parameter τ will ultimately be taken as x_4, or rather t. Treating all four coordinates x_i alike preserves the formal invariance of our results; it would also facilitate the discussion of certain special motions (e.g. linear accelerations).

Consider now an event $P(x_i)$ on the world line L of a particle A of our continuum. We denote the unit vector tangent to L at P by λ_i; λ_i being time-like, we have

$$\lambda_i \lambda_i = -1 , \qquad \lambda_i = \frac{\partial x_i}{\partial \tau}\left(-\frac{\partial x_k}{\partial \tau}\frac{\partial x_k}{\partial \tau}\right)^{-1/2} .$$

Let $Q(x_i + \delta x_i)$ be a neighbouring event on the world line M of an adjacent particle B. We can express the normal distance between L and M in terms of λ_i and the displacement δx_i from P to Q (see Fig. 2A). We write

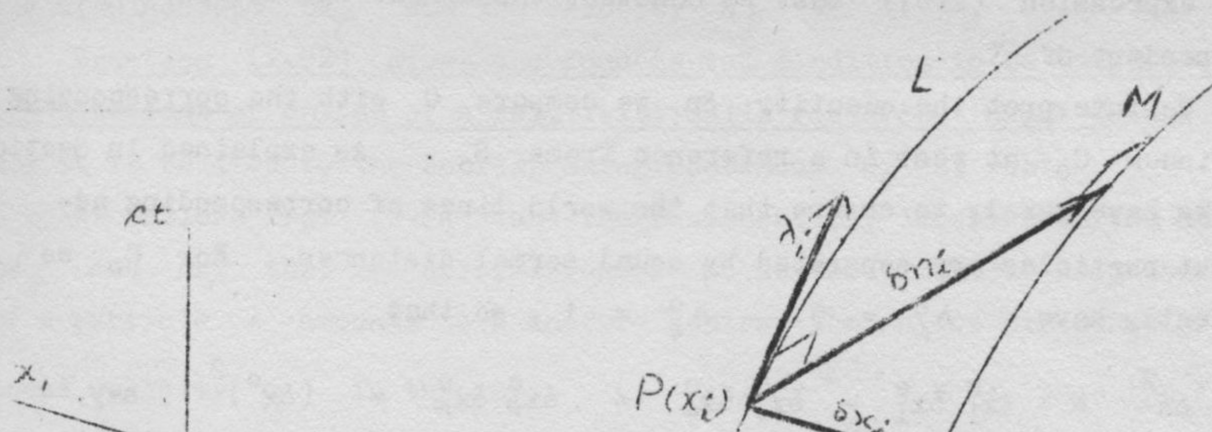

Fig. 2A: Two rigidly connected particles in space-time

δx_i as the sum of two orthogonal displacements: $\lambda_i \, \delta\theta$ parallel to L, and δn_i perpendicular to L (and therefore space-like). Thus

$$\delta x_i = \delta n_i + \lambda_i \, \delta\theta \, ,$$

$$0 = \lambda_i \, \delta n_i \, .$$

Hence

$$\lambda_i \, \delta x_i = - \, \delta\theta \, ,$$

$$\delta x_i \, \delta x_i = \delta n_i \, \delta n_i - \delta\theta^2 \, .$$

By eliminating $\delta\theta$ we therefore get for the infinitesimal distance δn between the world lines L and M

$$\delta n^2 = \delta n_i \, \delta n_i = \delta x_i \, \delta x_i + (\lambda_i \, \delta x_i)^2 \, . \qquad (2.01)$$

If the continuum C is rigid, then for every pair of adjacent particles this expression (2.01) must be constant throughout the motion, i.e. independent of τ .

To interpret the quantity δn we compare C with the <u>corresponding</u> continuum C_0 at rest in a reference frame S_0 . As explained in Section 1 , we have merely to ensure that the world lines of corresponding adjacent particles are separated by equal normal distances. For C_0 we evidently have $\lambda^0_\rho = 0$, $\lambda^0_4 = i$, so that

$$\delta n^2 = \delta x^0_i \, \delta x^0_i - \delta x^0_4 \, \delta x^0_4 = \delta x^0_\rho \, \delta x^0_\rho = (\delta x^0)^2 \quad \text{, say,}$$

where δx^0_ρ is the displacement vector between the particles A and B in the continuum C_0 ; $\delta n = \delta x^0$ is therefore the distance between these particles.

Let us suppose that in one particular reference system S the co-ordinates x_ρ of the particles have been expressed as functions of t , which is now identified with the parameter τ . We may take $\delta x_4 = 0$, so that δx_i is identified with δx_ρ , the instantaneous displacement vector from A to B in the continuum C . Denoting the velocity vector $\dfrac{\partial x_\rho}{\partial t}$ by u_ρ , with $u^2 = u_\rho u_\rho$, we then have, since $\dfrac{\partial x_4}{\partial t} = ic$,

$$\lambda_\rho = u_\rho (c^2 - u^2)^{-1/2} \quad , \qquad \lambda_4 = ic\,(c^2 - u^2)^{-1/2} \quad .$$

Thus equation (2.01) becomes

$$(\delta x^o)^2 = \delta x_\rho \, \delta x_\rho + (c^2 - u^2)^{-1} (u_\rho \, \delta x_\rho)^2 \quad , \tag{2.02}$$

which can be written in the forms

$$(\delta x^o)^2 = \delta x^2 \left(1 + \frac{u^2 \cos^2 \theta}{c^2 - u^2} \right) \quad , \tag{2.03}$$

$$(\delta x^o)^2 = (\delta x \sin \theta)^2 + \frac{(\delta x \cos \theta)^2}{1 - u^2/c^2} \quad , \tag{2.04}$$

where $\delta x = (\delta x_\rho \, \delta x_\rho)^{1/2}$ is the instantaneous distance between the particles A and B in the continuum C , and θ is the angle between the displacement δx_ρ and the velocity u_ρ .

<u>Equation (2.02) gives the fundamental condition to be satisfied by the equations of motion of a relativistically rigid continuum C</u> and shows how it is related to the corresponding continuum C_o at rest.

It follows from (2.04) and the fundamentally linear relation between δx_ρ and δx^o_ρ that the deformation of C in the immediate neighbourhood of a particle A amounts to a uniform contraction in the direction of the local velocity u_ρ in the ratio $(1 - u^2/c^2)^{1/2} : 1$; thus <u>the Fitz-Gerald-Lorentz contraction hypothesis is valid for infinitesimal elements of a rigid continuum</u>.

By applying (2.02) in turn to the sides of the triangles formed by <u>three</u> adjacent particles A , B , $\bar{B}$ in the continua C and C_o it is easy to deduce a relation between the corresponding angles χ , χ_o subtended at A by the other two particles. The distances from A to B and to $\bar{B}$ being δx and $\delta \bar{x}$ in C , δx^o and $\delta \bar{x}^o$ in C_o , we get

$$\delta x^o \, \delta \bar{x}^o \cos \chi_o = \delta x \, \delta \bar{x} \left(\cos \chi + \frac{v \, \bar{v}}{c^2 - u^2} \right) \quad , \tag{2.05}$$

where v and $\bar{v}$ are the components of the velocity u_ρ (by orthogonal projection) along AB and $A\bar{B}$ respectively (see Fig. 2B); this supplements equation (2.03), by which the ratios of δx, $\delta\bar{x}$ to δx^0, $\delta\bar{x}^0$ are

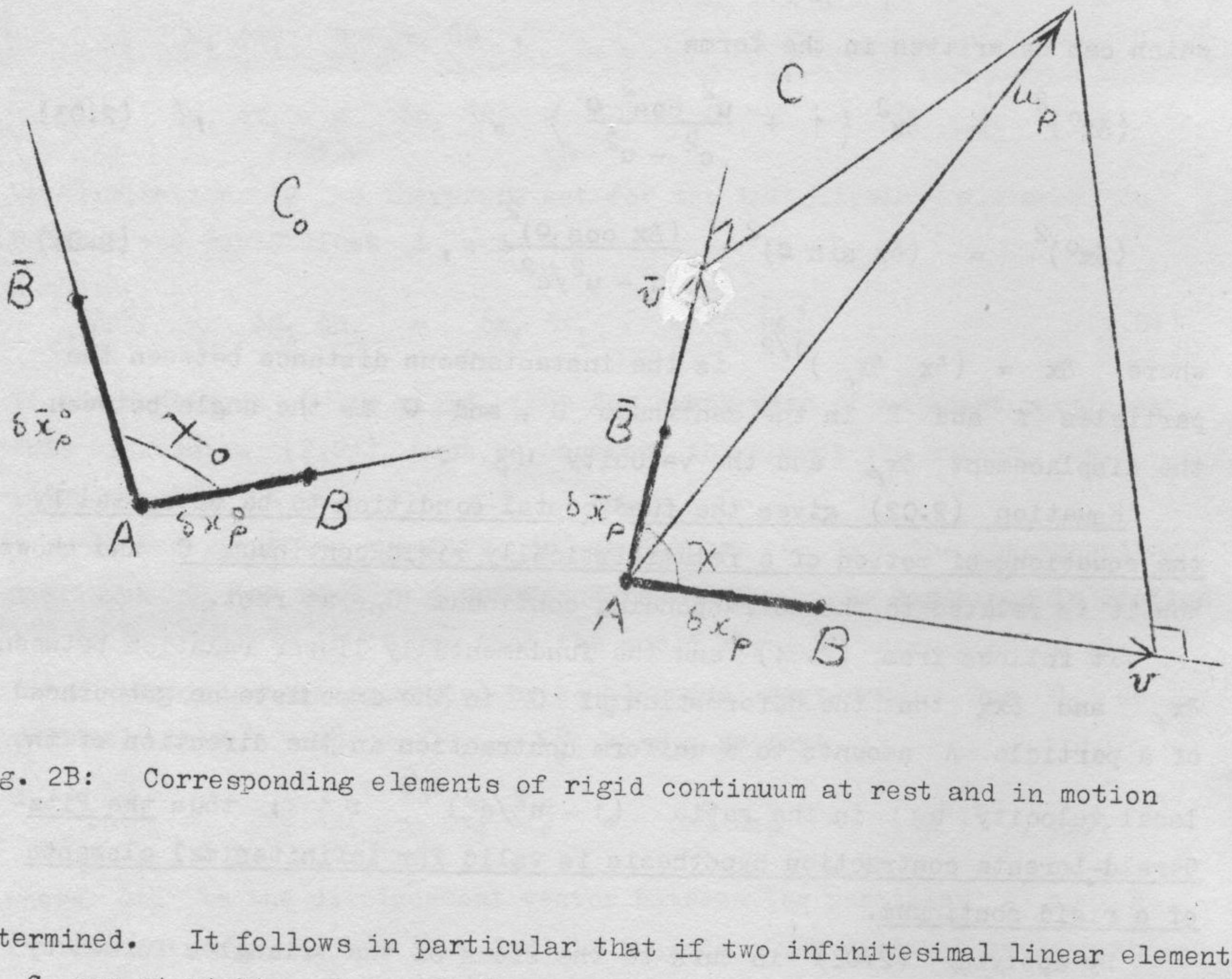

Fig. 2B: Corresponding elements of rigid continuum at rest and in motion

determined. It follows in particular that if two infinitesimal linear elements of C are at right angles and either of them is perpendicular to the velocity, then the corresponding elements in C_o are also at right angles (as might be inferred directly from the "local" FitzGerald-Lorentz contraction).

The dimension of the continuum C being N (where $N = 1, 2,$ or 3), independent variation of the Lagrangian parameters a_R distinguishing the particles determines, at each point of C and C_o, N parametric lines. (Capital indices take the values 1 to N.) To infinitesimal elements of each of these lines we can apply equation (2.03), and to each <u>pair</u> we can apply equation (2.05); there are thus $N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$ independent conditions to be satisfied by the functions $x_\rho(a_1, \ldots, a_N;\ t)$. <u>These</u>

relations, obtainable by putting $\delta x_\rho = \sum' \frac{\partial x_\rho}{\partial a_R} \delta a_R$, $\delta x^o_\rho = \sum' \frac{\partial x^o_\rho}{\partial a_R} \delta a_R$, directly in (2.02) and equating coefficients in the resulting quadratic identity in the δa_R, take the form of non-linear partial differential equations, already given by Synge (loc. cit.):

$$g^o_{RS}\,(a_1, \ldots a_N) = \frac{\partial x_\rho}{\partial a_R}\frac{\partial x_\rho}{\partial a_S} + (c^2 - u^2)^{-1}\left(u_\rho \frac{\partial x_\rho}{\partial a_R}\right)\left(u_\sigma \frac{\partial x_\sigma}{\partial a_S}\right). \quad (2.06)$$

Here the g^o_{RS} $\left(= \frac{\partial x^o_\rho}{\partial a_R}\frac{\partial x^o_\rho}{\partial a_S}\right)$ are the coefficients of the line-element in the continuum C_o, and are independent of t; in agreement with the original definition of rigidity, they also give the metric in the neighbourhood of a particle $A\,(a_1, \ldots a_N)$ whenever its velocity vanishes (i.e. as observed in its instantaneous rest frame).

If $N = 3$ we are led by the six equations (2.06) to the Born rigid body, with its restricted freedom of motion. From now on we shall take $N = 2$, so that (2.06) gives three differential equations to be satisfied by the three functions $x_\rho\,(a_1, a_2;\, t)$ (with $u_\rho = \frac{\partial x_\rho}{\partial t}$) describing a relativistically rigid surface σ, and relates σ to the corresponding rigid surface σ_o at rest.

For such a surface we can deduce from (2.03), or directly from the FitzGerald-Lorentz contraction rule, that in the neighbourhood of a particle $A\,(a_1, a_2)$ the surface σ is contracted relative to σ_o in the ratio

$$\left(1 - \frac{v_t^{\,2}}{c^2 - v_n^{\,2}}\right)^{1/2} : 1 \qquad (2.07)$$

in the direction of v_t, the tangential component of velocity; here v_n is the component normal to σ, with $u^2 = v_n^{\,2} + v_t^{\,2}$. In this connection we note that $v_t^{\,2}$ is expressible in terms of v and $\bar{v}$, the quantities so denoted in (2.05), viz. the orthogonal projections of the velocity u_ρ on any two lines of σ making an angle χ with each other (see Fig. 2C) :

$$v_t^2 = \operatorname{cosec}^2 \chi (v^2 + \bar{v}^2 - 2 v \bar{v} \cos\chi).$$

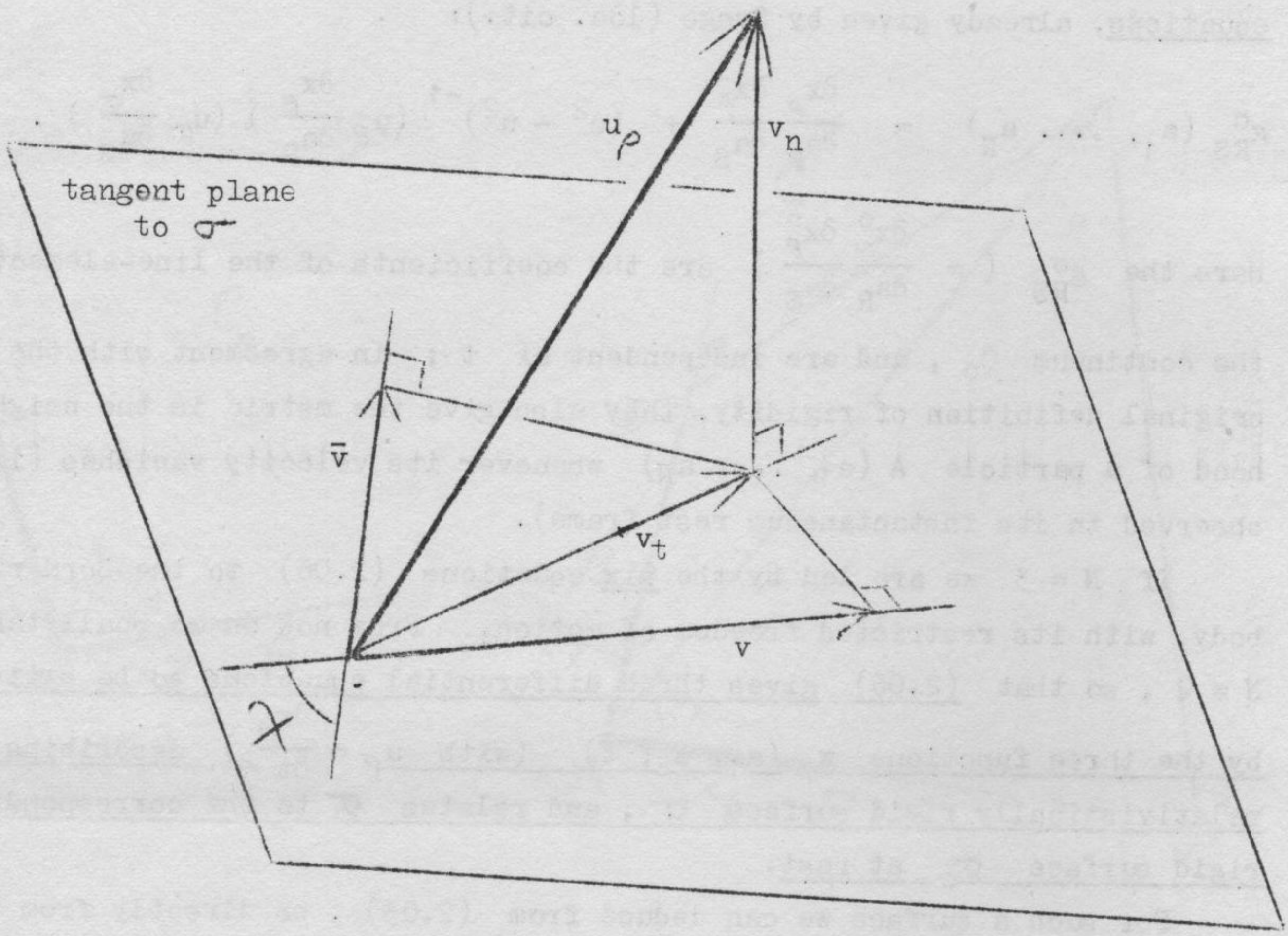

Fig. 2C: Velocity components on a surface

The condition that a linear element δx_ρ at any point (a_1, a_2) should have at a particular instant the same length on σ as it had on σ_0 is that it should be perpendicular to the velocity:

$$u_\rho \ \delta x_\rho \equiv u_\rho \frac{\partial x_\rho}{\partial a_1} \delta a_1 + u_\rho \frac{\partial x_\rho}{\partial a_2} \delta a_2 = 0 . \qquad (2.08)$$

Regarding the motion as known we can interpret this as a differential equation for the finite curves whose lengths are unchanged; they are the orthogonal trajectories on σ of curves given by projecting u_ρ on σ.

In order to determine the form of the moving surface σ when its initial form σ_0 at rest is given, we shall usually have to apply the conditions of rigidity in the form (2.06). Our next task is to see how these partial differential equations simplify when σ is taken to be a surface of revolution in symmetrical motion.

3. Surface of revolution in symmetrical motion

In this section the conditions for superficial rigidity will be applied to a moving surface σ that appears to a certain Galileian observer S to be permanently symmetric about a fixed line, its <u>axis</u>, its motion being axially symmetric as well; σ may change its shape from instant to instant. Under these assumptions it is easily seen that with no loss of generality the equations of motion of σ can be written in the form

$$\begin{aligned} x_1 + i\,x_2 &= r\,(s,\,t)\,e^{i\theta}, \\ x_3 &= z\,(s,\,t) \end{aligned} \tag{3.01}$$

where the x_3-axis has been taken along the axis of symmetry. Here

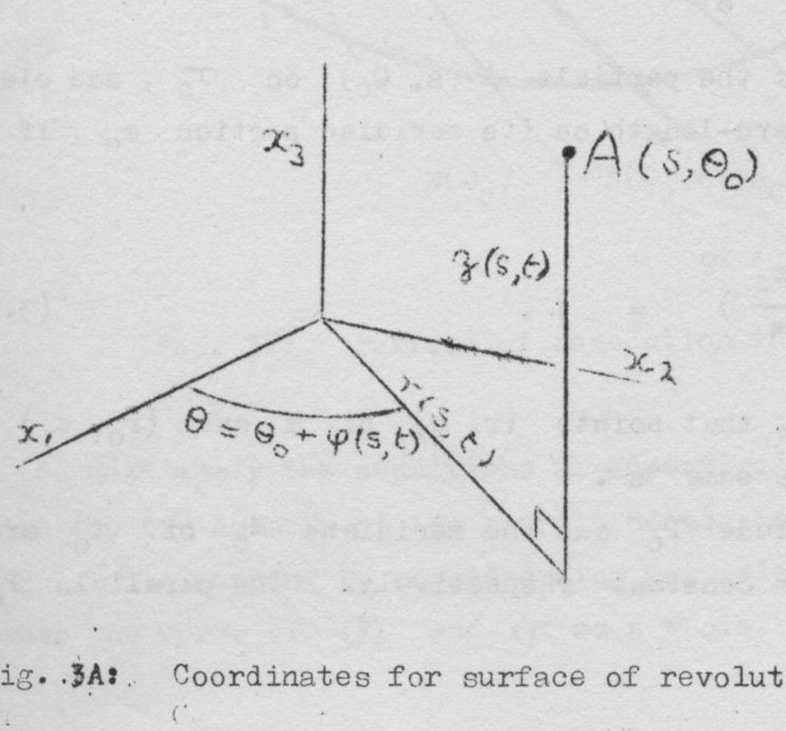

Fig. 3A: Coordinates for surface of revolution in symmetrical motion

$\theta = \theta_0 + \varphi(s, t)$, s and θ_0 being the Lagrangian labels of a particle A of σ ; r , z and θ are cylindrical polar coordinates (see Fig. 3A). The shape of σ is determined at any instant t by its <u>meridian section</u> m , which is a curve in an r,z-plane with parametric equations

$$r = r(s, t), \quad z = z(s, t) \qquad (3.02)$$

The parallels of latitude P on σ are given by $s = \text{constant}$; they are <u>permanent</u> in the sense that the particles (s, θ_0) forming P at one instant form another parallel P' at another instant. The meridians M are given at any instant by $\theta_0 + \varphi(s, t) = \text{constant}$, and are evidently <u>not</u> permanent in general.

The corresponding surface σ_0 at rest in a reference frame S_0 is also assumed to be a surface of revolution, as is indeed implied by the conditions of rigidity; so far as its correspondence with σ is concerned we may suppose S_0 and S to coincide, and likewise the two axes of symmetry (any Lorentz transformation required to bring this about preserves <u>all</u> Minkowskian distances, and therefore the property of rigidity). The surface σ_0 can therefore be described by equations (3.01) at the instant $t = t_0$ say, when the velocities all vanish. By redefining the parameter θ_0 if necessary, we can assume that $\varphi(s, t_0) = 0$; denoting $r(s, t_0)$ and $z(s, t_0)$ by $r_0(s)$ and $z_0(s)$, we get as the equations of σ_0 :

$$\begin{aligned} x_1^0 + i\,x_2^0 &= r_0(s)\, e^{i\theta_0} , \\ x_3^0 &= z_0(s) . \end{aligned} \qquad (3.03)$$

Thus θ_0 is the azimuth of the particle $A(s, \theta_0)$ on σ_0 , and clearly s may be interpreted as the arc-length on its meridian section m_0 , if we add the condition

$$\left(\frac{dr_0}{ds}\right)^2 + \left(\frac{dz_0}{ds}\right)^2 = 1 . \qquad (3.04)$$

We shall say, by extension, that points (r, z) of m and (r_0, z_0) of m_0 <u>correspond</u> if they have the same s .

The parallels of latitude P_0 and the meridians M_0 of σ_0 are given by $s = \text{constant}$ and $\theta_0 = \text{constant}$ respectively. The parallels P_0 are mapped at each instant into corresponding (i.e. having the same s) parallels P of σ, whereas the meridians M_0 are mapped not into meridians but into twisted curves N on σ all congruent to one another, by symmetry, and having the equation $\theta = \varphi(s, t) + \text{constant}$ (see Fig. 3B).

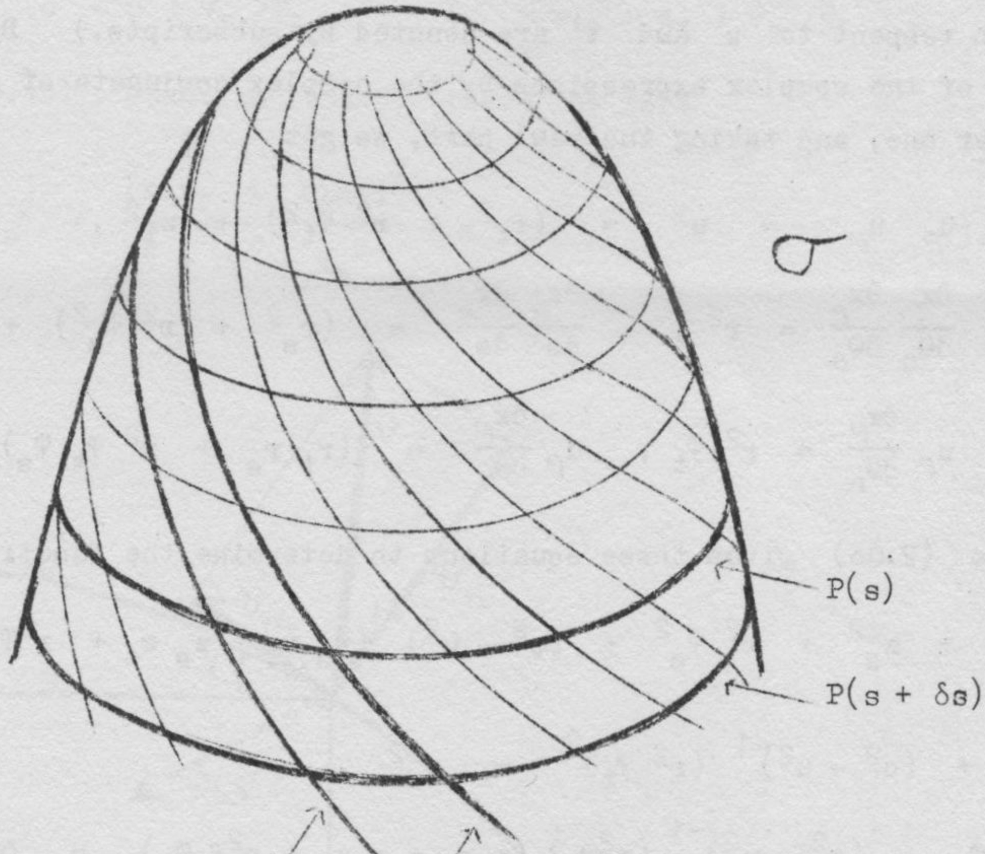

Fig. 3B: Surface of revolution in symmetrical motion

We next apply the conditions of superficial rigidity (2.06) to describe the two aspects of the correspondence between σ_0 and σ : first the pointwise mapping by corresponding particles, and then the relation between the forms of σ_0 and σ as a whole, i.e. between corresponding parallels of latitude P_o and P and between the meridian sections m_o and m. We put s and θ_o for a_1 and a_2 respectively. For the velocity components $u_\rho = \frac{\partial x_\rho}{\partial t}$ we have from (3.01)

$$u_1 + i u_2 = (r_t + i r \varphi_t) e^{i\theta} ,$$

$$u_3 = z_t ,$$

with similar expressions for the partial derivatives with respect to the Lagrangian labels s and θ_o. (Here and elsewhere partial derivatives with respect to s and t are denoted by subscripts.) By multiplying one of the complex expressions by the complex conjugate of itself or another one, and taking the real part, we get

$$u_\rho u_\rho = u^2 = (r_t^2 + r^2 \varphi_t^2) + z_t^2 ,$$

$$\frac{\partial x_\rho}{\partial \theta_o} \frac{\partial x_\rho}{\partial \theta_o} = r^2 , \qquad \frac{\partial x_\rho}{\partial s} \frac{\partial x_\rho}{\partial s} = (r_s^2 + r^2 \varphi_s^2) + z_s^2 ,$$

$$u_\rho \frac{\partial x_\rho}{\partial \theta_o} = r^2 \varphi_t , \quad u_\rho \frac{\partial x_\rho}{\partial s} = (r_t r_s + r^2 \varphi_t \varphi_s) + z_t z_s ;$$

thus (2.06) gives three equations to determine the functions r, z, and φ:

$$r_s^2 + z_s^2 + r^2 \varphi_s^2 + (c^2 - u^2)^{-1} (r_s r_t + z_s z_t + r^2 \varphi_s \varphi_t)^2 = 1 , \tag{3.05}$$

$$r^2 + (c^2 - u^2)^{-1} (r^2 \varphi_t)^2 = r_o^2 , \tag{3.06}$$

$$r^2 \varphi_s + (c^2 - u^2)^{-1} (r^2 \varphi_t)(r_s r_t + z_s z_t + r^2 \varphi_s \varphi_t) = 0 . \tag{3.07}$$

These equations emphasize the distortion of material elements of σ relative to the corresponding elements of σ_o, as explained at length in Section 2. But now the relation between corresponding elements δs of the meridian <u>sections</u> m and m_o can be isolated from the simultaneous <u>twisting</u> of σ (indicated by the curves N), in the following way: on substituting for u^2, equation (3.06) gives

$$\frac{\varphi_t^2}{c^2 - r_t^2 - z_t^2} = \frac{1}{r^2} - \frac{1}{r_o^2} , \tag{3.08}$$

and equation (3.07) can similarly be written

$$\varphi_t\,(r_s r_t + z_s z_t + r^2\varphi_s\varphi_t) + \varphi_s\,(c^2 - u^2) = 0 , \qquad (3.09)$$

or

$$\varphi_t\,(r_s r_t + z_s z_t) + \varphi_s\,(c^2 - r_t^{\,2} - z_t^{\,2}) = 0 . \qquad (3.10)$$

The last two terms on the left hand side of (3.05) are, by (3.09), equal to

$$r^2\varphi_s^{\,2} + (c^2 - u^2)\,\varphi_s^{\,2}/\varphi_t^{\,2} = (c^2 - r_t^{\,2} - z_t^{\,2})\,\varphi_s^{\,2}/\varphi_t^{\,2} .$$

Thus, from (3.10), equation (3.05) takes the form

$$r_s^{\,2} + z_s^{\,2} + \frac{(r_s r_t + z_s z_t)^2}{c^2 - r_t^{\,2} - z_t^{\,2}} = 1 . \qquad (3.11)$$

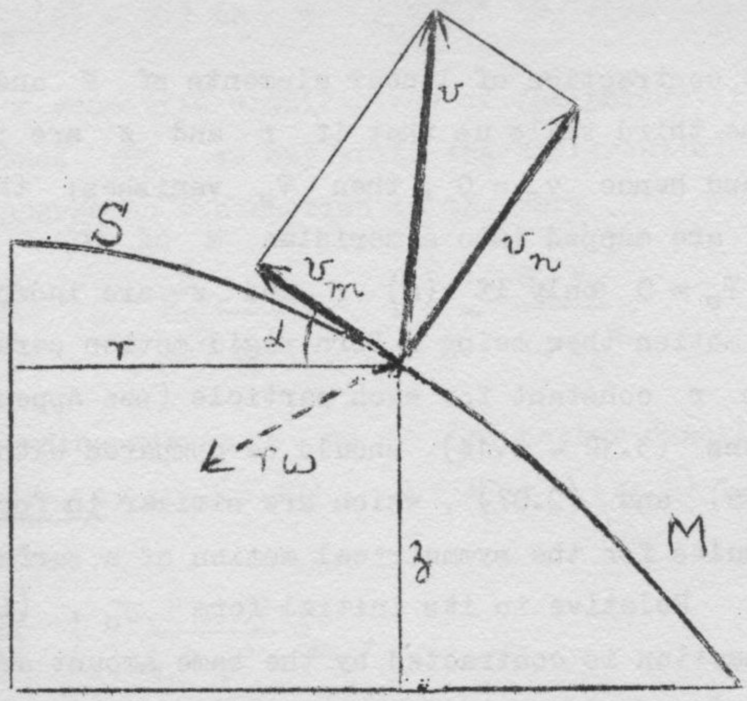

Fig. 3C: Velocity components in meridian plane

The relations (3.11), (3.08), and (3.10) can easily be interpreted geometrically (see Fig. 3C). Let

S be the arc-length along the meridian M (or m), with $S_s^2 = r_s^2 + z_s^2$;

v, the component of velocity in the meridian plane, with $v^2 = r_t^2 + z_t^2$;

v_n, the component of velocity normal to σ, with $S_s v_n = r_s z_t - z_s r_t$;

v_m, the component of velocity along M , with $S_s v_m = r_s r_t + z_s z_t$;

and $r\omega = r\,\varphi_t$, the component of velocity along the parallel P .

Then the equations above may be written in the form

$$S_s^2 = \frac{c^2 - v^2}{c^2 - v_n^2} = 1 - \frac{v_m^2}{c^2 - v_n^2}, \tag{3.12}$$

$$\frac{1}{r^2} - \frac{1}{r_o^2} = \frac{\omega^2}{c^2 - v^2}, \quad \text{or} \quad \left(\frac{r}{r_o}\right)^2 = \frac{c^2 - u^2}{c^2 - v^2} = 1 - \frac{\omega^2 r^2}{c^2 - v^2}, \tag{3.13}$$

$$\frac{\varphi_s}{\varphi_t} = - \frac{S_s\, v_m}{c^2 - v^2} = \pm \left(\frac{1 - S_s^2}{c^2 - v^2}\right)^{1/2}. \tag{3.14}$$

The first two give the contraction of linear elements of M and P relative to M_o and P_o . The third tells us that if r and z are independent of t , so that $v = 0$, and hence $v_m = 0$, then φ_s vanishes; that is, the particles forming M_o are mapped into a meridian M of σ. Conversely, it can be shown that $\varphi_s = 0$ only if (a) r and z are independent of t , or (b) $\varphi_t = 0$, the motion then being a Born rigid motion parallel to the axis of symmetry, with r constant for each particle (see Appendix A).

The three relations (3.12 - 3.14) should be compared with the general relations (2.03 - 2.05) and (2.07) , which are similar <u>in form</u>. We can then summarize our results for the symmetrical motion of a surface of revolution σ as follows: Relative to its initial form σ_o , (i) each element of the meridian section is contracted by the same amount as it would be if it were a material element and there were <u>no</u> axial rotation; (ii) each parallel of latitude contracts according to the FitzGerald-Lorentz contraction rule for surfaces, remaining a parallel of latitude; (iii) each parallel of latitude is rotated as a whole, in addition to shrinking radially, i.e. the surface σ is twisted about its axis.

In the exceptional case where r and z depend on s only, so that v and v_m vanish, it follows from (3.12) that $S_s = 1$, and from (3.13) and (3.14) that ω is a pure constant. Thus the meridian m is uncontracted relative to m_o, there is no twist, and so the meridians M are <u>permanent</u> and of fixed shape. This special case will be treated in Section 4.

Finally we note the condition for no contraction of a material linear element of our surface of revolution: according to equation (2.08) such an element $(\delta\theta_o, \delta s)$ has at the instant t the same length on σ as on σ_o if

$$(r_s r_t + z_s z_t + r^2 \varphi_s \varphi_t)\, \delta s + r^2 \varphi_t\, \delta\theta_o = 0 ,$$

which, by means of (3.09) and (3.13), can be rewritten as

$$\varphi_s (c^2 - u^2)\, \delta s = r^2 \omega^2\, \delta\theta_o ,$$

or

$$\varphi_s (c^2 - v^2)\, \delta s = r_o^{\,2} \omega^2\, \delta\theta_o . \qquad (3.15)$$

Recalling the geometric significance of the parameters s and θ_o on the initial surface σ_o, we may write the differential equation for the curves Γ_o that undergo no contraction in the form

$$\frac{d\theta_o}{ds} = \frac{c^2 - v^2}{\omega^2 r_o^{\,2}} \varphi_s = - \frac{S_s v_m}{\omega r_o^{\,2}}. \qquad (3.16)$$

The corresponding curves Γ on σ are better described in terms of θ and s, where, as in (3.10), $\theta = \theta_o + \varphi(s, t)$; equation (3.15) then gives

$$\frac{d\theta}{ds} = \frac{c^2 - v^2}{\omega^2 r^2} \varphi_s ,$$

or, in terms of S, the arc-length along M (t of course remaining fixed),

$$\frac{d\theta}{dS} = \frac{c^2 - v^2}{\omega^2 r^2} \frac{\varphi_s}{S_s} = - \frac{v_m}{\omega r^2} . \qquad (3.17)$$

(In (3.16) and (3.17), the alternative expressions are obtained from the relations (3.13) and (3.14).) It follows that the angle between Γ

and M is not less than that between Γ_o and M_o. Since the right-hand sides of these equations involve only s, the surfaces σ_o and σ are covered at each instant by an infinity of congruent curves Γ_o and Γ.

These results will be used in Section 5 in dealing with uniform screw motion.

4. Uniform rotation about axis of symmetry

In Section 3 we applied the conditions of rigidity to a surface of revolution σ whose axis of symmetry was fixed relative to a particular observer, but which could change its form from instant to instant. The relations obtained there are still too complicated to yield solutions of our fundamental problem, and we may seek to simplify them by restricting the motions to be rigid in the Newtonian sense. As can be seen from equations (3.06), (3.07), and (3.05), the Newtonian condition $\delta x = \delta x^o$ requires that r, φ_s, and z_s should each be independent of t; by (3.11) and (3.08), the same must be true of z_t and φ_t, which are therefore constants. Thus σ must be in uniform screw motion along its axis; we shall first suppose that the axial velocity component vanishes, postponing the general case to Section 5.

We consider then the uniform rotation of a surface of revolution σ with angular velocity ω. This is described by the equations (cf. (3.01))

$$\begin{aligned} x_1 + i\,x_2 &= r\,(s)\,e^{i(\theta_o + \omega t)} \,, \\ x_3 &= z\,(s)\,, \end{aligned} \tag{4.01}$$

where, as before, θ_o is the azimuth and s the meridian arc-length on the initial surface σ_o. Since the velocity is now wholly circumferential, the conditions of rigidity (3.12 - 3.14) reduce to two equations for r and z, relating the meridian section m of σ to the section m_o of the initial surface σ_o at rest:

$$r_s^{\,2} + z_s^{\,2} = S_s^{\,2} = 1\,, \tag{4.02}$$

$$\frac{1}{r^2} - \frac{1}{r_o^{\,2}} = \frac{\omega^2}{c^2}\,. \tag{4.03}$$

These equations are fundamental in all the following work of this and later sections. The second of them can be written (cf. (3.13))

$$\frac{r}{r_0} = \left(1 + \frac{\omega^2 r_0^2}{c^2}\right)^{-1/2} = \left(1 - \frac{\omega^2 r^2}{c^2}\right)^{1/2} , \qquad (4.04)$$

and yields on differentiation

$$\frac{dr}{dr_0} = \left(1 + \frac{\omega^2 r_0^2}{c^2}\right)^{-3/2} = \left(1 - \frac{\omega^2 r^2}{c^2}\right)^{3/2} . \qquad (4.05)$$

Thus the elements of the meridian section m_0 are not contracted by the rotation; the meridians M and M_0 now consist of the same particles and have the same length, $2\,\ell_0$, say: As in the more general case, the parallels of latitude, also formed of the same particles on σ as on σ_0, undergo the FitzGerald-Lorentz contraction due to the rotation.

The problem of determining either of σ and σ_0 when the other is given is evidently now reduced to the evaluation of an integral for $z(s)$ or $z_0(s)$, obtained by substituting from (4.05) into (4.02); this will be done for particular surfaces σ_0 in Sections 6 and 7. But without actually integrating, we can make several deductions from the relations above. We first make some abbreviations, needed again in later sections: we denote the products of ω/c with r, r_0, z, z_0, and s by the corresponding barred symbols, and introduce α and α_0, the inclinations of the tangents of the meridian sections m and m_0 to the equatorial plane (see Fig. 3C), between which there are such relations as

$$\cos\alpha = r_s = \frac{d\bar{r}}{d\bar{s}}, \qquad \sin\alpha = -z_s = -\frac{d\bar{z}}{d\bar{s}} .$$

In this notation equations (4.04) and (4.05) become

$$\bar{r}_0 = \bar{r}\,(1 - \bar{r}^2)^{-1/2}, \quad \text{or} \quad \bar{r} = \bar{r}_0\,(1 + \bar{r}_0^2)^{-1/2} , \qquad (4.06)$$

$$\cos\alpha = \cos\alpha_0\,(1 - \bar{r}^2)^{3/2} = \cos\alpha_0\,(1 + \bar{r}_0^2)^{-3/2} ; \qquad (4.07)$$

thus

$$z_s = -\sin\alpha = \pm\left[1 - \cos^2\alpha_0\,(1 + \bar{r}_0^2)^{-3}\right]^{1/2} . \qquad (4.08)$$

From these relations it is easy to infer from the mere existence of the initial surface σ_0 certain characteristics of the rotating surface σ, particularly as the angular velocity ω is taken larger and larger:

(i) According to (4.06), $\bar{r} \leq 1$; in other words the circumferential velocity of σ nowhere exceeds the speed of light, however large ω may be, although for any finite r_0 we have $\omega r \longrightarrow c$ as $\omega \longrightarrow \infty$.

(ii) According to (4.07), $\cos\alpha$ cannot exceed $(1 - \bar{r}^2)^{3/2}$, which approaches zero with increasing ω. In particular, α and z_s cannot vanish except on the axis, having in fact lower bounds given by (4.08). We may equally say that the curve of $\bar{s}$ against $\bar{r}$ (or of s against r) has at every point a slope exceeding that of the function $\bar{r}_0(\bar{r})$ of (4.06). Thus $z_s \Longrightarrow \pm 1$ as $\omega \longrightarrow \infty$, and the shape of σ therefore approaches that of a needle of diameter $2c/\omega$ and length $2\ell_0$.

A difficulty arises if the slope is of different signs on neighbouring arcs of the meridian section m, since it must not vanish anywhere between; this question, which is particularly acute for multiply-connected surfaces, will be discussed in Section 8.

(iii) The preceding remark (ii) implies that if the slope of m vanishes at the pole $(r = 0)$, then its curvature there, κ, is bounded below. In fact differentiating (4.07) twice yields the relation

$$\kappa^2 = \kappa_0^{\,2} + \frac{3\omega^2}{c^2}, \tag{4.09}$$

where κ_0 is the polar curvature on σ_0; thus $|\kappa|$ is at least $\sqrt{3}\,\omega/c$. If α is not zero at $r = 0$, the slopes and curvatures of m and m_0 are equal there.

We may summarize our principal results for a surface of revolution σ rotating with constant angular velocity ω as follows: Relative to the initial surface σ_0 1) the meridian arc-length is unaltered; 2) there is a radial contraction of amount $(1 - r^2\omega^2/c^2)^{1/2} : 1$, or equivalently $1 : (1 + r_0^{\,2}\omega^2/c^2)^{1/2}$, where r_0 and r are the initial and contracted distances from the axis; 3) the parallels of latitude are not twisted about the axis relatively to one another; and 4) the limiting form of σ as $\omega \to \infty$ is that of a needle of diameter $2c/\omega$ and length equal to that of the initial meridian section.

We shall next see how these conclusions are modified when the rotating surface has in addition a constant velocity along its axis of symmetry.

5. Uniform screw motion along axis of symmetry

We shall continue to designate by σ a rigid surface of revolution rotating with constant angular velocity ω about its axis in a Galileian reference frame S ; and we shall denote by Σ the congruence of world lines of its particles in space-time, their equations being taken in the form (4.01) , supplemented by the relation $x_4 = ict$. From this motion can be derived others not essentially different, simply by taking the point of view of other Galileian observers, whose descriptions of the motion and the consequent deformation relative to a corresponding surface at rest will depend on their own motion relative to S . We shall show in this section that a uniform screw motion of a surface of revolution along its axis can be so derived (although of course it could equally well be deduced from the general theory of Section 3 by appropriate specialization).

The congruence of world lines Σ consists of helices whose axes coincide with the x_4-axis; the world lines of particles lying on one parallel of latitude P of σ (s = constant) generate a circular cylinder of radius r(s) in the 3-flat $x_3 = z(s)$. We wish to intersect this congruence with the 3-flat $t' =$ constant, where t' denotes the time as measured in a Galileian frame of reference S' having a velocity $-U$ in the direction of the x_3-axis. We therefore take the new coordinates x'_i to be related to those of S by the Lorentz transformation

$$\begin{aligned} x'_1 + i\,x'_2 &= x_1 + i\,x_2 , \\ x'_3 &= \gamma(x_3 + U t) , \\ t' &= \gamma(t + U x_3 / c^2) , \end{aligned} \tag{5.01}$$

where $\gamma = (1 - U^2/c^2)^{-1/2}$, the last two relations being equivalent to

$$x'_3 = x_3/\gamma + U t' , \qquad t = t'/\gamma - U x_3 / c^2 .$$

Under this transformation the congruence Σ given by (4.01) takes the form

$$x'_1 + i\,x'_2 = r'(s)\,e^{i\,(\theta' + \omega' t')}\ , \qquad (5.02)$$

$$x'_3 = z'(s) + U\,t'\ , \qquad (5.03)$$

$$x'_4 = ict'\ , \qquad (5.04)$$

with

$$r' = r(s)\ , \quad z' = z(s)\,/\,\gamma\ , \quad \theta' = \theta_0 - \frac{\omega U}{c^2}\,z(s)\ , \quad \text{and} \quad \omega' = \omega/\gamma. \qquad (5.05)$$

We shall denote the congruence of world lines whose equations are expressed in the form (5.02 - 5.04) by Σ', although it is of course identical with Σ. The surface whose history in space-time is Σ', and whose motion in the frame of reference S' is given by (5.02 - 5.03), will then be denoted by σ'.

We note first that the motion of any particle of σ' is compounded of 1) a uniform rotation round the x'_3-axis with angular velocity ω', and 2) a uniform axial translation with speed U. Secondly, the form of σ' at the instant $t' = 0$ is given by

$$x'_1 + i\,x'_2 = r'(s)\,e^{i\theta'}\ ; \qquad (5.06)$$

$$x'_3 = z'(s)\ , \qquad (5.07)$$

and is therefore a surface of revolution with meridian section m' obtained from the meridian section m of σ, according to (5.05), by a uniform axial contraction in the ratio $1 : \gamma$; the particles forming a meridian M given by $\theta_0 = 0$, say, now compose the uniformly twisted curve N' given by

$\theta' = -\dfrac{\omega U}{c^2}\,z = -\dfrac{\omega'\gamma^2 U}{c^2}\,z'$ on σ'. (A snapshot of σ' at any other time t' would have the same form as at $t' = 0$, since the increase in azimuth θ' and in x'_3 would be the same for all particles.) The correspondence between uniform rotation and uniform screw motion has now been established: the histories of the two motions in space-time can be made to coincide provided the angular velocities are taken in the ratio

<u>$1 : (1 - U^2/c^2)^{1/2}$, where U is the axial velocity in the screw motion</u>.

Throughout the remainder of this section we shall adopt the point of view of the observer S' and regard the quantities ω' and U (or γ) as the two fundamental parameters determining the motion of the rigid surface σ' , and therefore its deformation relative to the initial surface σ_0. It is important to bear in mind, however, that only the product $\gamma\omega' = \omega$ has an absolute significance, and we shall calculate the effect on σ' of varying ω' or γ by first recalling the effect on σ of the corresponding change in ω and then superposing the two effects of the axial velocity U established above (see Fig. 5).

The effect of varying ω' , with γ fixed, is easy to describe. As ω' (and therefore ω) increases, the radius of σ' shrinks, and the length approaches the finite value $2\,\ell_0/\gamma$, where $2\,\ell_0$ is the length of the initial meridian; at the same time the twist increases, the number of complete turns in the curve N' being approximately $\frac{\gamma U}{\pi c} \cdot \frac{\omega' \ell_0}{c}$ when ω' is large. (It should be emphasized that in this discussion we are not envisaging linear or angular <u>accelerations</u>; the words "varying", "approaching", etc. do not refer to time.)

More interesting, though more complicated, is the behaviour of σ' as U is varied <u>with ω' fixed</u>, particularly as $\gamma \longrightarrow \infty$. We begin by disregarding the twist about the axis and concentrating on the relation between the meridian sections m' and m_0 . The relation between m' and m is given by the formulas $r' = r$, $z' = z/\gamma$, where r and z are determined completely in terms of the coordinates r_0, z_0 of m_0 by the relations (4.02 - 4.03) (with which we may associate (4.06 - 4.08)). The parameter ω occuring there is to be replaced by $\omega'\gamma$, and so varies when γ does.

As γ increases, then, the axial contraction of m' is uniquely combined with the radial contraction of m given by (4.04): according to remark (ii) of Section 4, the ultimate diameter and length of σ (as $\omega \to \infty$) are $2c/\omega$ and $2\,\ell_0$, so that the ultimate diameter and length of σ' (as $\gamma \longrightarrow \infty$) are $2c/(\omega'\gamma)$ and $2\,\ell_0/\gamma$. Moreover the ultimate diameter is the same for all points on the meridian initially at a finite distance from

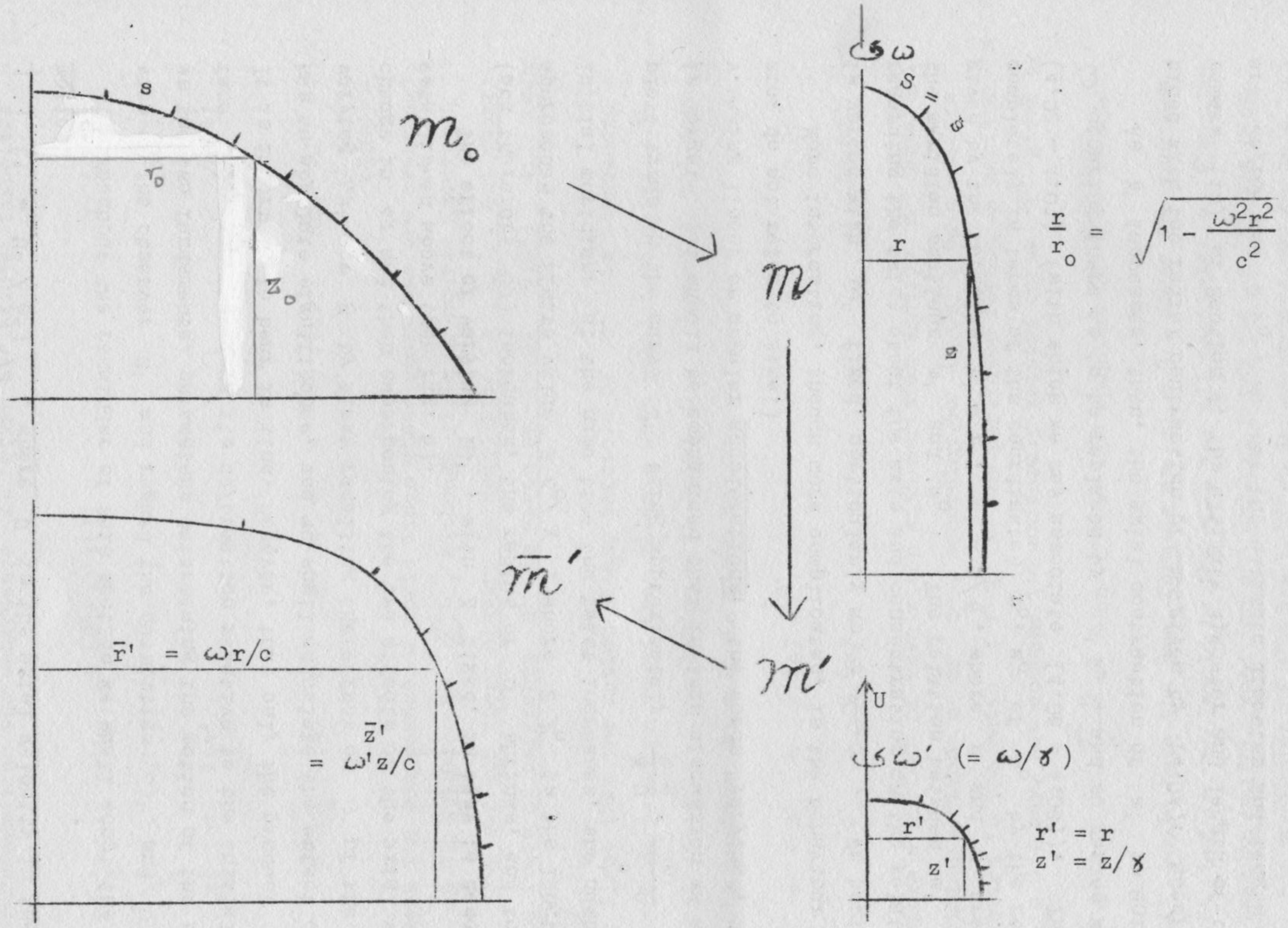

Fig. 5A: Distortion of meridian section due to uniform screw motion the axis. The ratio of the maximum dimensions has ultimately the <u>finite</u> value $\omega' \ell_0/c$, and so depends only on the observed angular velocity and the length of the meridian of the initial surface σ_0 .

More precisely, it can be shown that <u>whatever may be the form of σ_0 , the ultimate shape of σ' as $\gamma \to \infty$, ω' remaining finite, is a circular cylinder with flat ends</u>, its dimensions being those just given. To see this it is sufficient to study the dependence on γ of the slope and curvature of the meridian section m' . The detailed investigation is described in Appendix B; a summary of the results appears below. Here we shall merely show that, when γ is large, the slope of m' is finite only on small arcs for which the radius is near its maximum value, and that the corresponding arcs on m_0 are very small and lie near the axis. In the first place, we have from (4.04)

$$r' = \frac{c}{\omega}\left(1 + \frac{c^2}{\omega^2 r_0{}^2}\right)^{-1/2} ,$$

so that r' will differ from $c/(\omega'\gamma)$ by only a small amount provided $c/(\omega r_0)$ is small, i.e. provided $c/(\omega' r_0)$ is $\mathcal{O}(\gamma)$. This allows r_0 to be of order $\gamma^{-1+\epsilon}$ $(\epsilon > 0)$, and so arcs of σ_0 near the axis can correspond to arcs of σ' where $r' \sim c/\omega'\gamma$. Now if m' has a finite slope, $\frac{dr'}{ds}$ and $\frac{dz'}{ds} = \frac{1}{\gamma}\frac{dz}{ds}$ must be of the same order in γ ; they must therefore be of order γ^{-1} , since, by (4.02) , $\left(\frac{dr}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$.
But since from (4.05) we have

$$\frac{dr}{ds} = \frac{dr_0}{ds}\left(1 + \frac{\omega'^2\,\gamma^2}{c^2} r_0{}^2\right)^{-3/2} ,$$

it follows that $\omega' r_0/c$ is of order $\gamma^{-2/3}$. It can be seen also that if $\omega' r_0/c$ is of lower (higher) order in γ , then the slope of m' will be small (large) and the corresponding arc will have $r' < \frac{c}{\omega'\gamma}$ $\left(r' \sim \frac{c}{\omega'\gamma}\right)$.

For the more detailed investigation of the curvature it is convenient to consider, instead of m' , the <u>finite</u> curve $\bar{m}'$ obtained from it by magnification in the ratio $\omega/c : 1$, i.e. $\omega'\gamma/c : 1$ (see Fig. 5). The coordinates $\bar{r}'$, $\bar{z}'$ of $\bar{m}'$ are related to the barred coordinates of Section 4 by the formulas

$$\bar{r}' = \bar{r}, \qquad \bar{z}' = \bar{z}/\gamma, \tag{5.08}$$

where we recall that

$$\bar{r}_0 = \omega r_0/c, \qquad \bar{r} = \omega r/c, \qquad \bar{z} = \omega z/c.$$

The limiting value of $\bar{r}'$ is accordingly unity (cf. remark (i) of Section 4). The results of the calculations in Appendix B are summarized in the following table, and confirm the statement made above about the ultimate shape of σ' as its axial speed U is increased while its angular velocity ω' is kept fixed. These conclusions will be verified later in particular cases (see Sections 6 and 7).

$\omega' r_0/c$	$O(\gamma^{-1})$	$O(\gamma^{-4/5})$	$O(\gamma^{-2/3})$	$O(\gamma^{-1/2})$	Finite
$\bar{r}_0 = \gamma\omega' r_0/c$	Finite	$O(\gamma^{1/5})$	$O(\gamma^{1/3})$	$O(\gamma^{1/2})$	$O(\gamma)$
$\bar{r}' = \gamma\omega' r'/c$	< 1	~ 1	~ 1	~ 1	~ 1
$1 - \bar{r}'$	Finite	$O(\gamma^{-2/5})$	$O(\gamma^{-2/3})$	$O(\gamma^{-1})$	
Slope of $\bar{m}'$	Small	$O(\gamma^{-2/5})$	Finite	$O(\gamma^{1/2})$	Large
Curvature of $\bar{m}'$	Small	Finite	$O(\gamma^{2/3})$	Finite	Small

Although the maximum dimensions of a surface of revolution σ' rotating with angular velocity ω' approach zero when it is moved along its axis with a speed U approaching that of light, nevertheless there are material curves on σ' whose lengths remain finite even in the limit; this is due to the extreme twist suffered by σ' when U is large.

Let us first study the curve N' on σ' into which a meridian M_o (or M) is mapped; according to (5.05), its equation can be written

$$\theta' = -\frac{\omega U}{c^2} z(s) = -\frac{U}{c}\gamma \bar{z}' . \qquad (5.09)$$

This is a spiral, or helix: while θ' increases by 2π radians, z' increases by $2\pi c^2/(\gamma^2 \omega' U)$, which approaches zero rapidly as $\gamma \longrightarrow \infty$; each parallel of latitude P' has been rotated by an angle proportional to its distance from the plane of some particular one. If β' denotes the angle between N' and M', and S' denotes the arc-length along M', then

$$\tan \beta' = r' \frac{d\theta'}{dS'} = \gamma \bar{r}' \frac{U}{c} \sin \alpha' ,$$

$\tan \alpha'$ being the slope of M'; for any finite r_o, $\tan \beta' \longrightarrow \gamma$ as $\gamma \longrightarrow \infty$, so that the helix N' makes a small constant angle with the parallels P'. If the slopes of the meridians vanish at the poles, then it is easy to see that $\tan \beta' \longrightarrow \sqrt{3}\, \bar{r}'^2$ there, so that N' is tangent to M' at the pole.

The element of length on this helix N' is given by the formula

$$(d\,\ell')^2 = \left(1 - \frac{U^2}{c^2} \sin^2 \alpha \,(1 - \bar{r}'^2)\right) ds^2 ,$$

which shows that N' and M_o have nearly the same length when U is near the speed of light.

Let us next find the material curves Γ' on σ' that have <u>exactly</u> the same arc-length as the <u>corresponding</u> curves Γ_o on σ_o. In accordance with the general equation (3.17), and keeping in mind our present use of primed symbols, we determine Γ' from

$$\frac{d\theta'}{ds} + \frac{U}{\omega r^2} \frac{dz}{ds} = 0 .$$

Thus by (5.08) the curve Γ' is inclined to the meridian M' at an angle $\tan^{-1} \left(\frac{U \gamma}{c \bar{r}'} \sin \alpha' \right)$, which for any finite r_o approaches $\pi/2 - \gamma^{-1}$

as $\gamma \longrightarrow \infty$; near the pole of σ' this angle approaches $\pi/3$, so that Γ' is there approximately an equiangular spiral. The corresponding curve Γ_0 is determined from

$$\frac{d\theta_0}{ds} + \frac{U}{\omega r_0^2}\frac{dz}{ds} = 0,$$

and is therefore inclined to the meridian M_0 at an angle of order γ^{-1}, except near the pole where it is a spiral similar to Γ' :

The length of these curves Γ_0 and Γ' is given by

$$\int_0^{2\ell_0} \left[1 + \left(\frac{U \sin\alpha}{\gamma\omega' r_0}\right)^2\right]^{1/2} ds,$$

taken along m_0. This evidently exceeds $2\ell_0$, the length of a meridian M_0, by very little if γ is very large.

We now leave the theory of a general surface of revolution in uniform screw motion and apply the results to particular surfaces.

6. Application to sphere

The shape of any surface of revolution σ in uniform rotation can be found from that of its initial form σ_0 at rest by evaluating the indefinite integral for $z(s)$ obtained from equations (4.02 - 4.03). The effect of the rate of rotation ω on such quantities as the axial dimensions of σ may be found by a process of power-series expansion and term-by-term integration, provided ω is sufficiently small; but for even moderately large rates of rotation it is necessary to consider special surfaces σ_0. In this section we take σ_0 to be a sphere of radius R_0 and find its distortion when made to rotate at constant speed ω. As Lagrangian labels we take the azimuth θ_0 and the colatitude α_0 (we need consider α_0 only in the range 0 to $\pi/2$). The meridian arc-length measured from the pole is $s = R_0\alpha_0$, and the coordinates of the meridian section m_0 are $r_0 = R_0 \sin\alpha_0$ and $z_0 = R_0 \cos\alpha_0$. We introduce as a fundamental parameter χ_0, where

$\omega R_0/c = \bar{R}_0 = \tan\chi_0$; then according to (4.02 - 4.03) the coordinates r , z of the meridian section m of the rotating surface σ are

$$r = R_0 \sin\alpha_0 \,(1 + \tan^2\chi_0 \sin^2\alpha_0)^{-1/2} , \qquad (6.01)$$

$$z = R_0 \int_{\alpha_0}^{\pi/2} \left[1 - \cos^2\alpha_0 \,(1 + \tan^2\chi_0 \sin^2\alpha_0)^{-3} \right]^{1/2} d\alpha_0 . \qquad (6.02)$$

The maximum radius of σ is $R = R_0 \cos\chi_0 = (c/\omega)\sin\chi_0$, so that the greatest velocity on σ is $c \sin\chi_0$; as $\omega \to \infty$, $\chi_0 \to \pi/2$. If ω is sufficiently small, direct expansion in power series gives

$$r = R_0 \sin\alpha_0 \,(1 - \tfrac{1}{2}\bar{R}_0^{\,2} \sin^2\alpha_0 + \tfrac{3}{8}\bar{R}_0^{\,4} \sin^4\alpha_0 - \dots) ,$$

$$z = R_0 \cos\alpha_0 \,(1 + \tfrac{1}{2}\bar{R}_0^{\,2} \cos^2\alpha_0 - \bar{R}_0^{\,4} (\cos^2\alpha_0 - \tfrac{3}{8}\cos^4\alpha_0) + \dots),$$

from which we get R and Z , the maximum values of r and z (attained at $\alpha_0 = \pi/2$ and 0 respectively):

$$R = R_0 \,(1 - \tfrac{1}{2}\bar{R}_0^{\,2} + \tfrac{3}{8}\bar{R}_0^{\,4} - \dots) ,$$

$$Z = R_0 \,(1 + \tfrac{1}{2}\bar{R}_0^{\,2} - \tfrac{5}{8}\bar{R}_0^{\,4} + \dots) , \qquad (6.03)$$

i.e.

$$Z/R_0 = 1 + (1 - R/R_0) - (1 - R/R_0)^2 + \dots$$

These expressions being useless for even moderate values of $\bar{R}_0$, we transform to a new variable x , lying in the range 0 to $\sin\chi_0$, such that

$$\tan\chi_0 \sin\alpha_0 = (\sin^2\chi_0 - x^2)^{1/2} (\cos^2\chi_0 + x^2)^{-1/2} . \qquad (6.04)$$

As shown in Appendix C, r and z are then given by

$$\bar{r} = \omega r/c = (\sin^2\chi_o - x^2)^{1/2} ,$$

$$\bar{z} = \omega z/c = \int_0^x \left[(1+x^2)^2 - x^2\sin^2\chi_o\right]^{1/2}(x^2+\cos^2\chi_o)^{-1}\,dx . \qquad (6.05)$$

The integrand in (6.05) can be expanded and integrated term by term for <u>any</u> value of ω, the result being $\bar{z} = I_o - \frac{1}{2}I_1 - \frac{1}{8}I_2 - \ldots$, where

$$I_n = \int_0^x (x\sin\chi_o)^{2n}(1+x^2)^{1-2n}(x^2+\cos^2\chi_o)^{-1}\,dx ; \qquad (6.06)$$

the first few of these are given in Appendix C. Putting $x = \sin\chi_o$, we find for Z the formula

$$\begin{aligned} Z/R_o = {} & \chi_o\sin\chi_o + \cos\chi_o - \frac{1}{2}\cot\chi_o\,(\tan^{-1}\sin\chi_o - \chi_o\cos\chi_o) \\ & - \frac{1}{8}\cot\chi_o\left[\left(\frac{1}{8} - \frac{5}{8}\cos^2\chi_o - \cot^2\chi_o\cos^2\chi_o\right)\tan^{-1}\sin\chi_o\right. \\ & \left. + \chi_o\cot^2\chi_o\cos\chi_o - \sin\chi_o\cos^2\chi_o\,(4+5\sin^2\chi_o)(1+\sin^2\chi_o)^{-2}\right] \\ & - \ldots . \end{aligned} \qquad (6.07)$$

When ω is large, i.e. χ_o near $\pi/2$, this can be shown to yield the asymptotic expression

$$Z = \frac{\pi}{2}R_o - \frac{c}{\omega}\frac{\pi}{8}\left(1 + \frac{1}{32} + \frac{3}{16.64} + \ldots\right) + \mathcal{O}\left(\frac{1}{\omega^3}\right) , \qquad (6.08)$$

$$\text{or} \quad Z/R_o = \frac{\pi}{2} - \frac{\pi}{8}\left(1 + \frac{1}{32} + \frac{3}{16.64} + \ldots\right)(R/R_o) + \mathcal{O}(R^3/R_o^3) .$$

The relation between Z and R given by these various expressions is shown in Fig. 6A as the curve $\gamma = 1$; as we should expect, R decreases from R_o to 0 and Z increases from R_o to $\frac{\pi}{2}R_o$ as ω increases from 0 to ∞ .

Before examining the shape of the meridian m in detail let us consider the special case where σ_o is a <u>plane</u>, i.e. a sphere of infinite radius. In terms of the variable x we have

$$\bar{r} = (1-x^2)^{1/2} , \qquad \bar{z} = \int_1^x \left[(1+x^2)^2 - x^2\right]^{1/2} x^{-2}\,dx , \qquad (6.09)$$

where the change in the lower limit signifies that z is now measured from the pole; the integral can be evaluated in terms of tabulated elliptic integrals by setting $x = \tan \psi/2$ (see Appendix C). When ω is large, $r \simeq c/\omega$ and we get a result for the surface σ analogous to (6.08):

$$- z = r_o + r \left[2\,E(1/2) - \frac{3}{2} K(1/2) \right] + \dots , \qquad (6.10)$$

where $E(k)$ and $K(k)$ are the standard complete elliptic integrals of modulus k ; the coefficient in square brackets may be verified to be the same as the coefficient of R/R_o in (6.08) . For <u>small</u> values of ω, direct expansion gives

$$- \bar{z} = \frac{\sqrt{3}}{2} \left(\bar{r}^{\,2} + \frac{1}{2} \bar{r}^{\,4} + \frac{7}{18} \bar{r}^{\,6} + \dots \right) , \qquad (6.11)$$

in agreement with remark (iii) of Section 4. The relation between $\bar{z}$ and $\bar{r}$ implied by (6.09 - 6.11) is of the form shown in Fig. 6B. These results will be used in the next section on the rotating cylinder.

Returning now to the finite sphere, we evaluate (6.05) by means of the elementary integrals I_n , the convergence being quite rapid; we thus have the meridian section m of the rotating surface σ corresponding to the original circular section m_o of the sphere σ_o .

As shown in Section 5, this calculation provides at the same time the shape of the meridian section m' of the corresponding surface of revolution σ' rotating with constant angular velocity $\omega' = \omega\gamma = \omega(1 - U^2/c^2)^{1/2}$ and moving along its axis with constant speed U : recalling that $r' = r$, we need only divide the function $z(\alpha_o)$ of (6.05) by γ to obtain the coordinates r' , z' of m' . The combined results of these calculations, depending on the two parameters γ and ω' , are presented in Figs. 6C, D, E.

The variation of the meridian section m' with ω' is shown in Fig. 6C: as $\omega' \longrightarrow \infty$, σ' becomes more and more needle-shaped, its ultimate length being $\pi R_o/\gamma$ (which equals the original diameter if $\gamma = \pi/2$).

The variation of m' with γ is shown in Fig. 6D: as $\gamma \longrightarrow \infty$, σ' resembles more and more a cylinder of height $\pi R_o/\gamma$ and diameter $2c/\gamma\omega'$ (which are equal if $\omega' R_o/c = 2/\pi$). The same curves magnified approximately in the ratio $\omega'\gamma/c : 1$ are shown in Fig. 6D .

The effect of γ on the maximum dimensions of σ' has been included in Fig. 6A, where curves γ = constant and ω' = constant are plotted in an R, Z-plane; the <u>axes</u> correspond to the needle-shaped and disk-shaped limiting forms of σ' as ω' and γ respectively approach infinity.

The twisting of σ' described in Section 5 has been suggested in Fig. 6C by indicating the curves N' on σ' that correspond to the original meridians M_o of σ_o .

7. <u>Application to circular cylinder</u>

We now take the surface of revolution σ_o to be a circular cylinder with flat ends, its radius being R_o and its height 2h . The problem of determining the corresponding surface of revolution σ in uniform rotation divides into two parts, the distortion of the mantle σ_{o1} and the flat end σ_{o2} being found independently. Continuity is ensured since z(s) is determined by (4.02) only to within a constant and r(s) is a continuous function of $r_o(s)$, according to (4.03) . We must note here the ambiguity always present in the sign of z ; it has no significance by itself, but must be taken into account when fitting the surfaces σ_1 and σ_2 , corresponding to σ_{o1} and σ_{o2} , together. By regarding the cylinder σ_o as the limit of surfaces with continuous tangents for which the slope of the meridian section m never changes sign (see Section 4) , we conclude that the ends of our cylinder should bulge <u>out</u> when it is made to rotate.

The shape of the mantle σ_1 , corresponding to σ_{o1} , is found at once: by (4.03) its radius r has the constant value $R = R_o (1 + \bar{R}_o^2)^{-1/2}$, where $\bar{R}_o = \omega R_o/c$, and by (4.08) we have $dz/ds = \sin\alpha = \pm 1$, so that the generators M_1 of σ_1 are straight lines parallel to the axis, of length 2h .

As for σ_2 , the rotating surface corresponding to the flat end σ_{o2} of the cylinder, we remark that the quantity R_o plays no <u>essential</u> role: σ_{o2} might be any plane region perpendicular to the axis of rotation. The determination of σ_2 has therefore been sufficiently described in the preceding section (see equations (6.09 - 6.11) and Fig. 6B). In particular we note that the maximum height h_2 of σ_2 is given for ω large by $h_2 = R_0 - 0.406\frac{c}{\omega} + \mathcal{O}\left(\frac{1}{\omega^2}\right)$ (see Appendix C). The meridian section m_2 is drawn in Fig. 7A for various values of $\bar{R}_0$. The curves have been shifted in the x_3-direction so as to give $z(R_0)$ the value zero, since we wish to fit the surface σ_2 to the mantle σ_1 already obtained above.

The addition of an axial velocity U to the uniform rotation so far considered results, as stated in Section 5, in an axial contraction in the ratio $1 : \gamma$ where $\gamma^2 = (1 - U^2/c^2)^{-1}$; thus we have merely to reduce the ordinates of our meridian m in this ratio to find the meridian section m' of the surface of revolution σ' in screw-motion with angular velocity $\omega' = \omega/\gamma$ and axial speed U. The effect of the two parameters γ and ω' on the shape of σ' is shown in Fig. 7B, where in fitting upper and lower caps σ'_2 to the mantle σ'_1 we assume $h = R_0$.

Fig. 7B shows the simultaneous radial and axial contraction that occurs as $\gamma \longrightarrow \infty$ with ω' fixed: the maximum height and diameter are ultimately $4R_0/\gamma$ and $2c/\gamma\omega'$.

By taking $h = 0$ the original cylinder σ_0 becomes a flat disk. It appears from Fig. 7B that its two surfaces separate when it rotates, and there is an additional flattening and twisting when an axial velocity is superimposed; these results are in qualitative agreement with those of Ives (1945).

8. Conclusions

In this paper we have shown that between corresponding elements of a moving rigid surface σ and its initial form σ_0 at rest there is a relation analogous to the FitzGerald-Lorentz contraction formula: infinitesimal elements of σ are contracted in the direction of the tangential velocity component v_t in the ratio $\left(1 - v_t^2/(c^2 - v_n^2)\right)^{1/2} : 1$, where v_n is the normal velocity component. For a surface of revolution σ moving symmetrically about its axis the differential relation between the meridian section m and the corresponding meridian section m_0 of σ_0 is the same as if the circumferential velocity component were absent, being given in fact by the rule above.

The determination of the form of σ when that of σ_0 is given has been shown to depend on a single quadrature in the case of surfaces of revolution rotating with constant angular velocity about their axes or in uniform screw motion along their axes. It was found necessary to assume that the normal to the surface σ_0 at rest was nowhere parallel to the axis (except at the pole), in order that the conditions of superficial rigidity should not contradict one another. The deformation of uniformly rotating spheres and cylinders has been computed in detail, the general predictions for such motion being confirmed: as the angular velocity approaches infinity, the surface σ becomes more and more needle-shaped, its ultimate length being equal to that of its initial meridian. In screw motion there is in addition an axial contraction and axial twist, and as the translational velocity approaches that of light the surface σ shrinks in all dimensions, its ultimate shape being that of a cylinder with flat ends.

The only simple generalization of these results seems to be to include a uniform translational velocity perpendicular to the axis of rotation; this would follow by applying a Lorentz transformation to the equations of motion given here (Section 4). Such a motion is of course relativistically rigid, though not rigid in the Newtonian sense; the surface would not be symmetrical about its axis but would be of elliptical cross-section.

It would be extremely useful to have solutions of our fundamental equations of superficial rigidity other than those in which the angular velocity is constant. It is easy to verify that it is only in that special case that the arc-length of the meridian section m is not contracted relative to that of m_0. It was this requirement that forced us to restrict the shape of σ_0. For example our theory cannot be applied to a torus, unless we admit the possibility of a rigid surface developing a sharp <u>edge</u> when it moves, and even then we must be allowed to choose the site for this edge on the surface in such a way that the torus will not break. If the edge persists throughout the transition from rest to the final state of motion, then it is easily seen that it must always consist of the same particles and cannot be chosen at will. It is quite possible that the edge may not be present except at instants when the angular acceleration vanishes, but this solution of the difficulty only replaces it by another, since the position of the edge on a <u>simply-</u>

connected surface would then be indeterminate.

In a sense our treatment of the rotating disk or cylinder with flat ends is illegitimate, since the normal to σ_0 is parallel to the axis over a finite surface; it may be questioned whether our decision to take the rotating surface as convex is physically satisfactory.

More general questions might be raised: for example it has been seen that the instantaneous form and velocity distribution determine the line-element of the initial surface and presumably its finite form, and they might therefore be regarded as given quantities characterizing a rigid surface. On the other hand, if two surfaces are given, either with or without a particular mapping of their particles, one might ask whether there exists a velocity distribution over one of them that would make it correspond relativistically to the other (regarded as at rest).

In applying such a theory as the present one to experiments carried out on the rotating earth, the absence of any interpretation of observations using accelerated measuring-rods and clocks will be felt at once. Despite many previous discussions of this problem, it is questionable whether an adequate solution can be provided by the Special Theory of Relativity.

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Appendix A: Conditions under which a surface of revolution moving symmetrically is not twisted

It was shown in Section 3 that, when a rigid surface of revolution σ_0 at rest is made to move symmetrically about its axis, the resulting surface of revolution σ is in general twisted about the axis, the meridians M_0 of σ_0 being mapped into twisted curves N on σ (see Fig. 3B). It was stated there that σ could remain untwisted only if (a) the radial and axial velocity components vanished (the motion then necessarily being a uniform rotation), or (b) the radial and rotational velocity components vanished (the motion then necessarily being a Born rigid motion parallel to the axis of symmetry). This statement will now be proved: we have to consider several possibilities in turn, and eliminate most of them by appealing to the conditions of superficial rigidity and the existence of the initial surface σ_0.

Since a meridian M_0 of σ_0 with equation θ_0 = constant is mapped into a curve N on σ with equation $\theta = \varphi(s, t)$ + constant (cf. (3.01)), we have to investigate under what circumstances φ will depend on t alone, so that $\varphi_s \equiv 0$. According to (3.14), $\varphi_s \equiv 0$ only if either the velocity component v_m along the meridian, or the component $r\,\varphi_t$ along the parallel of latitude, vanishes identically.

(a) Let us first assume that $v_m = 0$. This means that

$$r_s\, r_t + z_s\, z_t = 0 . \tag{A1}$$

The conditions of rigidity (3.11) and (3.08) then give

$$r_s^2 + z_s^2 = 1 , \quad \text{(A2)} \qquad \text{and} \qquad \frac{\varphi_t^2}{c^2 - v^2} = \frac{1}{r^2} - \frac{1}{r_0^2} , \quad \text{(A3)}$$

where r_0 depends on s only, and $v^2 = r_t^2 + z_t^2$. To satisfy (A2), (A1), let us put

$$r_s = \cos\alpha , \quad z_s = -\sin\alpha , \quad r_t = v \sin\alpha , \quad z_t = v\cos\alpha . \tag{A4}$$

Since $r_{st} = r_{ts}$ and $z_{st} = z_{ts}$, we find by differentiating (A4) that

$$v\,\alpha_s = 0\,, \quad \text{(A5)} \qquad \text{and} \qquad v_s + \alpha_t = 0\,. \quad \text{(A6)}$$

(i) Let us first suppose that (A5) is satisfied in virtue of $v = 0$, so that $r_t = 0$ and $z_t = 0$. (A1) and (A2) are then satisfied, and (A3) tells us that φ_t is a constant ω, so that σ is in uniform rotation.

(ii) Let us now assume that v is $\neq 0$. Then, from (A5), α must depend on t alone, and (A6) gives $v = V(t) - s\alpha_t$. According to (A4) we have

$$r = s\cos\alpha + a(t)\,, \qquad z = -s\sin\alpha + b(t)\,; \quad \text{(A7)}$$

the three functions of t thus introduced are connected by the relations

$$a'(t) = V(t)\sin\alpha\,, \qquad b'(t) = V(t)\cos\alpha\,. \quad \text{(A8)}$$

We can now write (A3) in the form

$$\frac{1}{r_o^{\,2}} = \frac{1}{(s\cos\alpha + a)^2} - \frac{\varphi_t^{\,2}}{c^2 - (V - s\alpha_t)^2}\,. \quad \text{(A9)}$$

The right hand side of (A9) is a rational function of s, and must be actually independent of t, i.e. equal to its value at any particular instant t_o. The resulting identity in s can be transformed into a polynomial identity, and therefore holds for every value of s, whether physically significant or not. It follows that the zeros, poles, and residues of the right hand side are pure constants. In particular, since the second denominator cannot have a double zero, we conclude that α and $a(t)$ must be constants. Assuming for the moment that α is not zero, we see from (A8) that $V(t) = 0$, and thus $v = V - s\alpha_t = 0$. But this is in contradiction with the assumption, made at the beginning of the paragraph, that $v \neq 0$. The alternative is that $\alpha = 0$, with $v = V(t)$; from (A3) we then have

$$\frac{1}{(s+a)^2} - \frac{1}{r_o^{\,2}} = \frac{\varphi_t^{\,2}}{c^2 - V^2} = C^2 \quad \text{(constant)}\,,$$

and, by differentiating with respect to s,

$$\frac{1}{(s+a)^3} - \frac{1}{r_o^3} \cdot \frac{dr_o}{ds} = 0 .$$

Since $\frac{dr_o}{ds}$ cannot be greater than 1, it follows that $r_o = s + a = r$ and that $C = 0$, i.e. $\varphi_t = 0$. This motion is a very singular one: the surface σ is plane, and moves normal to itself with arbitrary velocity $V(t)$ and without rotation. (The initial surface σ_o is also plane.)

(b) <u>We shall now assume that</u> $v_m \neq 0$ <u>, but that</u> $\varphi_t = 0$ <u>, so that</u> φ is in fact constant. From (3.08) we have $r = r_o(s)$, and from (3.11) we get a differential equation for $z(s, t)$ which can be written

$$c\, z_s = (c^2 - z_t^2)^{1/2} \frac{dz_o}{ds} . \tag{A10}$$

The standard methods of solution, in which s and t are treated on the same footing, are not so convenient here as the following: We take $z_t = V$ as an independent variable in place of t, and write $s = S$, $z(s, t) = Z(S, V)$ and $t = T(S, V)$. For (A10) we thus get

$$c\,(Z_S - V\,T_S) = (c^2 - V^2)^{1/2} \frac{dz_o}{dS} , \tag{A11}$$

with

$$Z_V - V\,T_V = 0 . \tag{A12}$$

Differentiating (A12) with respect to S, and (A11) with respect to V, we get

$$Z_{VS} - V\,T_{VS} = 0 , \tag{A13}$$

$$c\,T_S = V\,(c^2 - V^2)^{-1/2} \frac{dz_o}{dS} . \tag{A14}$$

Thus, from (A11),

$$Z_S = c\,(c^2 - V^2)^{-1/2} \frac{dz_o}{dS} . \tag{A15}$$

Integrating (A14) and (A15) with respect to S , we get

$$c\,T \;=\; V\,(c^2 - V^2)^{-1/2}\, z_o \;+\; c\,T_o(\tau)\,, \tag{A16}$$

$$Z \;=\; c\,(c^2 - V^2)^{-1/2}\, z_o \;+\; Z_o(\tau)\,, \tag{A17}$$

where τ is some function of V ; we choose τ so that $c^2 T_o'^2 - Z_o'^2 = 1$ (the dash indicating differentiation with respect to τ). Then, differentiating (A16) and (A17) with respect to V and applying (A12) , we see that $Z_o' = V\,T_o'$. Thus $(c^2 - V^2)^{-1/2} = T_o'$ and $V\,(c^2 - V^2)^{-1/2} = Z_o'$, and so

$$t \;=\; T_o(\tau) \;+\; \frac{Z_o'(\tau)}{c}\, z_o(s)\,, \tag{A18}$$

$$z \;=\; Z_o(\tau) \;+\; c\,T_o'(\tau)\, z_o(s)\,. \tag{A19}$$

Interpreting τ/c as the proper time along the world line with coordinates $Z_o(\tau)$, $T_o(\tau)$, we see that the motion given parametrically by (A18) and (A19) is the general Born motion in one dimension (parallel to the z-axis).

We can summarize the several alternative ways in which the meridians of σ_o are mapped into meridians of σ as follows:

$\varphi_s = 0$

- $v_m = 0$
  - $v = 0$
    - $\underline{\varphi_{tt} = 0}$ (uniform rotation)
  - $v \neq 0,\quad \alpha_s = 0$
    - $\underline{\alpha = 0,\ \varphi_t = 0,\ r = \text{const.}}$ (plane moving normally)
- $v_m \neq 0$
  - $\varphi_t = 0$
    - <u>Born motion</u>

Appendix B: Curvature of meridian section of surface of revolution in uniform screw motion

In this appendix we shall discuss, in more detail than in Section 5, the deformation of a surface of revolution σ' in uniform screw motion along its axis. The angular velocity ω' of the surface is to be kept fixed; we shall be especially interested in the form of σ' as the axial speed U approaches c , the speed of light. As before, the deformation of σ' relative to the initial surface σ_0 at rest will be found from that of the intermediate surface of revolution σ rotating uniformly with angular velocity $\omega = \gamma\omega'$, where $\gamma = (1 - U^2/c^2)^{-1/2}$.

The fundamental equations connecting the corresponding meridian sections m_0, m, and m' are (cf. (4.04), (4.02), (4.07), (5.05), and Fig. 5):

$$r = r_0 \left(1 + \frac{\omega^2 r_0^2}{c^2}\right)^{-1/2}, \qquad \cos\alpha_0 = \frac{dr_0}{ds},$$

$$\left(\frac{dr}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1, \qquad \cos\alpha = \frac{dr}{ds} = \frac{\cos\alpha_0}{(1 + \omega^2 r_0^2/c^2)^{3/2}},$$

$$r' = r, \qquad z' = z/\gamma, \qquad (\omega = \gamma\omega').$$

Here s is the arc-length along m_0 (and m) ; $r_0 = r_0(s)$; and α, α_0 are the angles made by the normals to m , m_0 with the axis of symmetry.

If we consider a definite parallel of latitude on σ_0 , i.e. fix s , then no matter what its initial radius r_0 may be, we shall have $\omega r/c \longrightarrow 1$ as $\gamma \longrightarrow \infty$, so that all the particles of σ_0 (except those at poles, where $r = r_0 = 0$) ultimately find themselves at the same distance from the axis. (This is the point of view adopted in Case C below.) On the other hand, for a <u>fixed</u> value of γ , no matter how large, there will be values of r intermediate between 0 and the maximum possible, c/ω . In order to study the shape of σ' (i.e. of m') we therefore fix γ at a conveniently large value and then investigate how the slope and curvature vary as r_0 is varied over the meridian section m_0 . (This is the point of view adopted in Cases A and B below.)

It turns out that by restricting γ to sufficiently large values we need consider mainly the neighbourhood of a pole $(r_0 = 0)$. For definiteness we shall assume that (a) $\alpha_0 = 0$ at the pole, so that σ_0 has a unique tangent plane there; and (b) the curvature κ_0 of σ_0 has the finite value κ_{00} at the pole and is a continuous function of s in its neighbourhood. Accordingly there exists a number δ such that for $r_0 < \delta$ we have

$$\alpha_0 \ll 1 , \qquad\qquad |\alpha_0 - \kappa_{00}\, r_0| \ll \kappa_{00}\, r_0 ,$$

from which it follows that

$$\kappa_{00}\,\delta \ll 1 . \tag{B1}$$

For $r_0 > \delta$ we make no assumptions about the curve m_0 other than that α_0 and κ_0 exist everywhere except possibly at isolated points.

We are now in a position to fix γ . As already stated, ω' (>0) is a given fixed quantity; in fact a fundamental dimensionless parameter in this discussion is $\omega'/(c\,\kappa_{00})$, for which however we may substitute $\omega'\delta/c$. We choose for γ a number at least so large that $\gamma\omega'\delta/c \gg 1$; in this way we guarantee that $\bar{r}_0 = \omega r_0/c$ can be large even though α_0 is sufficiently approximated by $\kappa_{00}\, r_0$. We may note here a consequence of our choice of γ : using (B1) we have

$$\gamma\omega'/c \gg \kappa_{00} ; \quad \text{or} \quad \bar{r}_0 \gg \kappa_{00}\, r_0 . \tag{B2}$$

We shall need exact expressions for the slopes p , p' and the curvatures κ , κ' of our meridian sections m , m' ; these are readily found to be

$$p = \tan\alpha , \quad \kappa = (1 + \bar{r}_0^{\,2})^{-3/2} \operatorname{cosec}\alpha \left(\kappa_0 \sin\alpha_0 + \frac{3\,\bar{r}_0\,\omega\cos^2\alpha_0}{c\,(1 + \bar{r}_0^{\,2})} \right) , \tag{B3}$$

$$p' = p/\gamma , \quad \kappa' = \frac{\kappa}{\gamma}\left(\cos^2\alpha + \frac{1}{\gamma^2}\sin^2\alpha\right)^{-3/2} = \frac{\kappa}{\gamma}\sec^3\alpha\,(1 + p'^2)^{-3/2} . \tag{B4}$$

In discussing the ultimate shape of σ' it is more convenient to consider instead of m' the finite curve $\bar{m}'$ obtained by magnifying m' in the ratio $\omega/c : 1$, i.e. $\gamma\omega'/c : 1$. The coordinates of a typical point on $\bar{m}'$ are $\bar{r}' = \bar{r}$, $\bar{z}' = \omega' z/c$; thus the maximum radius of $\bar{m}'$ approaches 1 as $\gamma \to \infty$ (see Fig. 5). The slope of $\bar{m}'$ is $\bar{p}' = p'$, but the curvature $\bar{\kappa}'$ is smaller than that of m', being in fact $\kappa' c / (\gamma \omega')$. Now by using (B1), (B2) it can be seen that, so long as $r_o < \delta$, the first term in the numerator of the expression (B3) for κ is negligible in comparison with the second, and we can therefore write

$$\bar{\kappa}' = \frac{1}{\gamma}\left(\frac{c\,\kappa}{\omega}\right) \sec^3\alpha \, (1 + p'^2)^{-3/2} , \tag{B5}$$

where

$$\frac{c\,\kappa}{\omega} \approx 3\,\bar{r}_o \operatorname{cosec}\alpha \, (1 + \bar{r}_o^{\,2})^{-5/2} \qquad (r_o < \delta) . \tag{B6}$$

On the other hand, since $\bar{r}_o \gg 1$ when $r_o > \delta$, we can write

$$\frac{c\,\kappa}{\omega} \approx \frac{1}{\gamma\,\bar{r}_o^{\,3}}\left(\frac{c\,\kappa_o}{\omega'}\sin\alpha_o + \frac{3\,c\cos^2\alpha_o}{\omega' r_o}\right) \qquad (r_o > \delta) , \tag{B7}$$

the bracket being independent of γ. By the symbol $\approx$ we mean that the relative error being made will approach zero as γ increases.

In writing (B7) we anticipated the fact that $\sin\alpha \approx 1$ when $r_o > \delta$. We can write such approximations for $\cos\alpha$ and $\sin\alpha$ in three exhaustive cases:

(A) If $r_o < \delta$ and $\bar{r}_o \ll 1$, then $\cos\alpha \approx 1 - \frac{3}{2}\bar{r}_o^{\,2}$ and $\sin\alpha \approx \sqrt{3}\,\bar{r}_o$; (B8)

(B) If $r_o < \delta$ and $\bar{r}_o \gg 1$, then $\cos\alpha \approx (\bar{r}_o)^{-3}$ and $\sin\alpha \approx 1$; (B9)

(C) If $r_o > \delta$, then $\cos\alpha \approx \cos\alpha_o\,(\bar{r}_o)^{-3}$ and $\sin\alpha \approx 1$. (B10)

Let us write $\omega' r_o/c = \rho_o$, so that $\bar{r}_o = \gamma\rho_o$. It follows from (B4 - B6) by using (B8) that in Case A, viz. $\bar{r}_o \ll 1$,

$$p' \approx \sqrt{3}\,\rho_o \quad \text{and} \quad \bar{\kappa}' \approx \sqrt{3}\,\gamma^{-1} . \tag{B11}$$

In Case B, viz. $\bar{r}_0 \gg 1$, we see from (B9) that

$$p' \approx \gamma^2 \rho_0^3 \quad \text{and} \quad \frac{c\kappa}{\omega} \approx 3\,\gamma^{-4}\,\rho_0^{-4}, \tag{B12}$$

but to estimate $\bar{\kappa}'$ we must distinguish two sub-cases according as $p' \gtrless 1$; then (B5) shows that

$$(B_1) \quad \text{if } p' \ll 1, \quad \text{then} \quad \bar{\kappa}' \approx 3\gamma^4 \rho_0^5, \tag{B13}$$

$$(B_2) \quad \text{if } p' \gg 1, \quad \text{then} \quad \bar{\kappa}' \approx 3\gamma^{-2} \rho_0^{-4}. \tag{B14}$$

Finally, in Case C, viz. $r_0 > \delta$, the same equations with (B10) show that

$$p' \approx \gamma^2 \rho_0^3 \sec\alpha_0, \tag{B15}$$

$$\bar{\kappa}' \approx \gamma^{-2} \rho_0^{-3} \left(\frac{c\,K_0 \sin\alpha_0}{\omega'} + \frac{3\cos^2\alpha_0}{\rho_0} \right); \tag{B16}$$

or we may say that $\bar{\kappa}'$ is $\mathcal{O}(\gamma^{-2})$.

Cases A and B are separated by values of ρ_0 of order γ^{-1}, for which $\bar{r}_0$ is finite, p' and $\bar{\kappa}'$ being $\mathcal{O}(\gamma^{-1})$, i.e. both small.

Cases B_1 and B_2 are separated, according to (B12), by values of ρ_0 of order $\gamma^{-2/3}$, for which p' is finite and $\bar{\kappa}'$ is $\mathcal{O}(\gamma^{2/3})$, i.e. large.

The curvature $\bar{\kappa}'$ is finite <u>twice</u> in Case B: first when ρ_0 is $\mathcal{O}(\gamma^{-4/5})$, p' being $\mathcal{O}(\gamma^{-2/5})$, i.e. small, and secondly when ρ_0 is $\mathcal{O}(\gamma^{-1/2})$, p' being $\mathcal{O}(\gamma^{1/2})$, i.e. large.

Ultimately, in Case C, p' is at least $\mathcal{O}(\gamma^2)$ and $\bar{\kappa}'$ is $\mathcal{O}(\gamma^{-2})$.

These conclusions are in agreement with the summary in Section 5.

Appendix C: Evaluation of Integrals in Section 6

This appendix supplies some of the mathematical steps omitted in Sections 6 and 7 on the rotating sphere and cylinder. Starting from the fundamental equations (6.01 - 6.02) for the coordinates r and z of the rotating surface σ, we introduce the variable x by the relation (cf. (6.04))

$$\tan^{-1}(x \sec \chi_o) = \sin^{-1}(\cos \alpha_o \sin \chi_o) = \chi .$$

In terms of x we have

$$\bar{r} = \frac{\omega r}{c} = (\sin^2 \chi_o - x^2)^{1/2} , \qquad (1 - \bar{r}^2) = \cos^2 \chi_o + x^2 ,$$

$$-\bar{r}\frac{d\alpha_o}{dx} = \cos \chi_o (\cos^2 \chi_o + x^2)^{-1} ,$$

$$\sin \alpha = \bar{r}\left[(1 + x^2)^2 - x^2 \sin^2 \chi_o\right]^{1/2} ,$$

so that the integral for z can be written

$$\bar{z} = \frac{\omega z}{c} = \int_0^x (1 + x^2)(\cos^2 \chi_o + x^2)^{-1}\left[1 - \frac{x^2 \sin^2 \chi_o}{(1 + x^2)^2}\right]^{1/2} dx , \quad \text{(C1)}$$

which is equivalent to equation (6.05).

By putting $x = \tan \psi/2$ we can express z in terms of incomplete elliptic integrals:

$$\bar{z} = \tan \frac{\psi}{2}(1 - k^2 \sin^2 \psi)^{1/2} + \frac{1}{2}\tan \chi_o \tan^{-1}\left(\frac{\tan \chi_o \sin \psi}{2\sqrt{(1 - k^2 \sin^2 \psi)}}\right)$$

$$- \int_0^\psi (1 - k^2 \sin^2 \psi)^{1/2} d\psi + \frac{1}{2}\int_0^\psi (1 - k^2 \sin^2 \psi)^{-1/2} d\psi$$

$$+ \frac{2 + \tan^2 \chi_o}{4}\int_0^\psi (1 - k^2 \sin^2 \psi)^{-1/2}(1 + k^2 \tan^2 \chi_o \sin^2 \psi)^{-1} d\psi ,$$

(C2)

where $k = \frac{1}{2} \sin \chi_o$. The elliptic integral of the third kind not being a tabulated function, the last term in (C2) would have to be expanded in series. That being so, it is just as convenient to return to (C1) and expand the square bracket by the binomial theorem. (The convergence in either case is at least as rapid as that of the series for $(1 - \frac{1}{4})^{1/2}$.) We thus get

$$\bar{z} = I_o - \frac{1}{2} I_1 - \frac{1}{8} I_2 - \frac{1}{16} I_3 - \cdots , \tag{C3}$$

where

$$I_n = \int_0^x (x \sin \chi_o)^{2n} (1 + x^2)^{1-2n} (\cos^2 \chi_o + x^2)^{-1} dx .$$

In particular we have

$$I_o = x + \chi \sin \chi_o \tan \chi_o ,$$

$$I_1 = \tan^{-1} x - \chi \cos \chi_o ,$$

$$\begin{aligned} I_2 = {} & \left(\frac{1}{8} - \frac{5}{8} \cos^2 \chi_o - \cot^2 \chi_o \cos^2 \chi_o \right) \tan^{-1} x \\ & + \chi \cot^2 \chi_o \cos \chi_o \\ & + \left(\frac{1}{8} - \frac{5}{8} \cos^2 \chi_o \right) \frac{x}{1 + x^2} - \frac{\sin^2 \chi_o}{4} \frac{x}{(1 + x^2)^2} . \end{aligned}$$

By putting $x = \sin \chi_o$ in (C3) we get formula (6.07) for Z , the height of the pole above the equatorial plane.

The analogous formulas for a plane can be obtained by letting $R_o \longrightarrow \infty$ in either (C1) or (C2) . It is convenient to measure z from the pole $r = 0$; the result is

$$\bar{z} = \int_{\psi}^{\pi/2} (4 - \sin^2 \psi)^{1/2} \operatorname{cosec}^2 \psi \, d\psi = H(\psi) \text{ , say.} \tag{C4}$$

$H(\psi)$ can be expressed as

$$H(\psi) = \cot\psi\,(4 - \sin^2\psi)^{1/2} + \int_{\psi}^{\pi/2} (4 - \sin^2\psi)^{1/2}\,d\psi - 3\int_{\psi}^{\pi/2} (4 - \sin^2\psi)^{-1/2}\,d\psi ,$$

or as a series corresponding somewhat to (C3) :

$$H(\psi) = 2\cot\psi + (2\psi - \pi)\frac{1}{8}\left(1 + \frac{1}{32} + \frac{3}{16 \cdot 64} + \dots\right) - \sin 2\psi\left(\frac{1}{256} + \frac{1}{2048} + \dots\right) + \dots \qquad (C5)$$

We are particularly interested in the behaviour of these various expressions as $\omega \longrightarrow \infty$. For example, from equation (6.07) , on putting $\cos\chi_0 = \delta = R/R_0$, we get the expansion (6.08) ; and similarly from (C5) we deduce that if ψ is small, $\bar{r}_0$ large, then

$$z \sim r_0 \left[1 - \frac{\pi}{8\,\bar{r}_0}\left(1 + \frac{1}{32} + \frac{3}{16 \cdot 64} + \dots\right) + \dots\right]$$

$$\approx r_0\,(1 - 0{\cdot}406 / \bar{r}_0) .$$

The equality of the coefficients of the terms linear in (c/ω) in these expansions for the sphere and the plane can be established directly by taking the appropriate limits in the integral (C1) .

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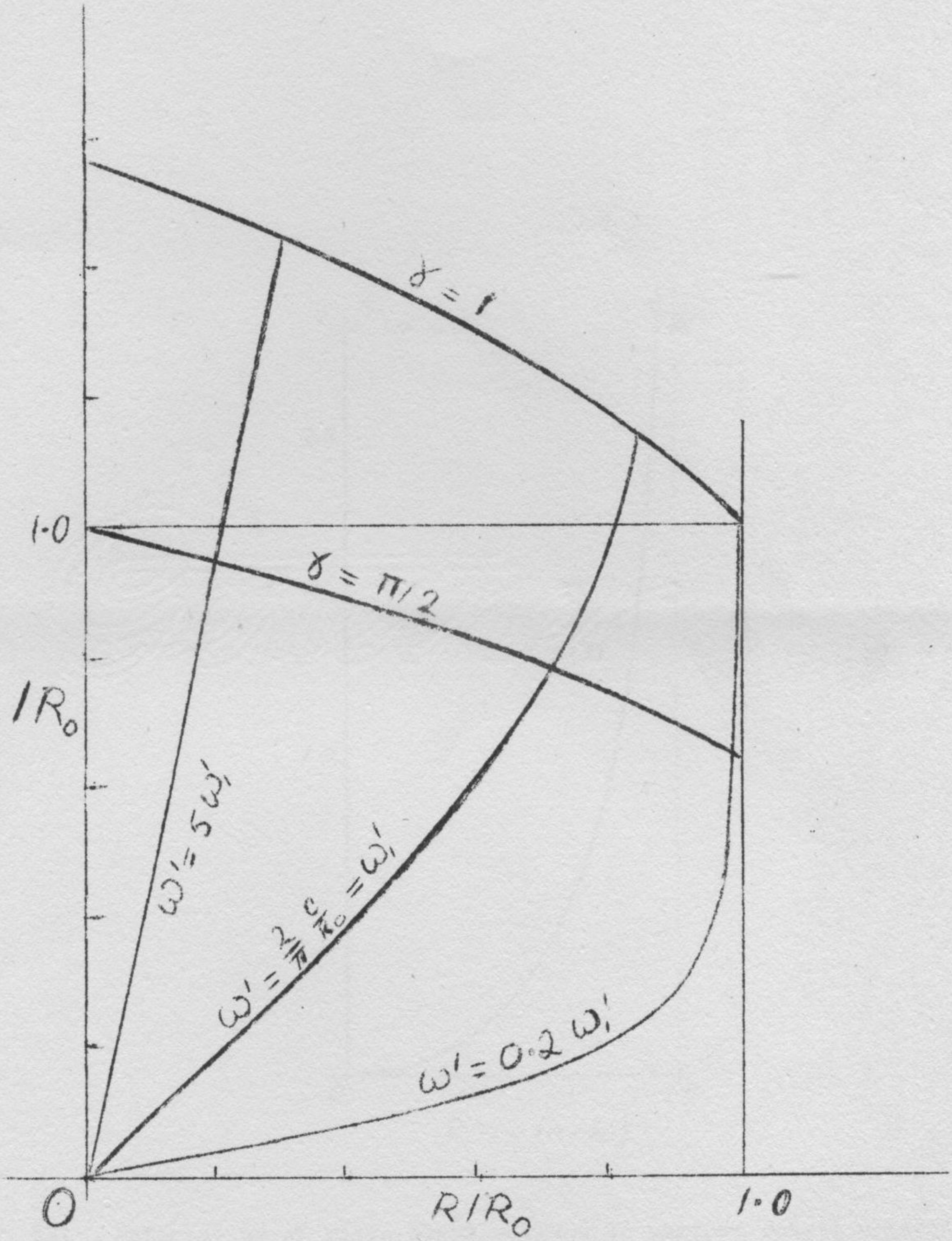


Fig. 6A: Maximum height Z and radius R of rigid sphere in uniform screw motion with speed U and spin ω' . $\gamma = (1 - U^2/c^2)^{-1/2}$.

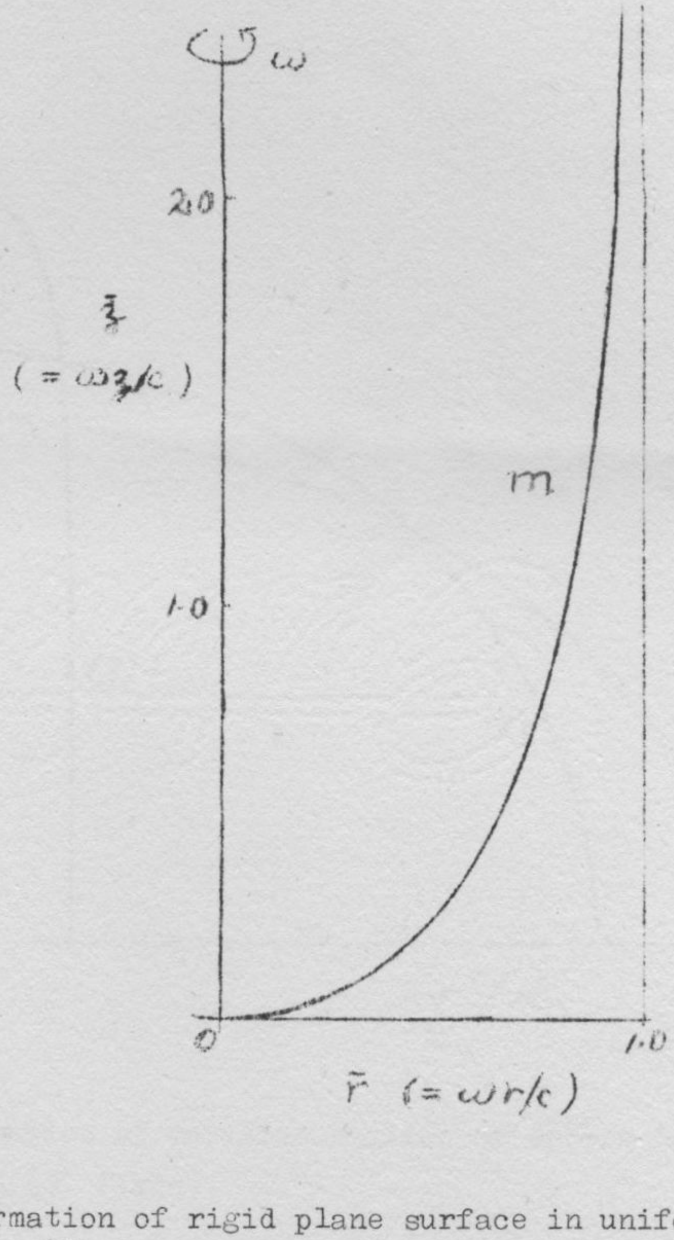


Fig. 6B: Deformation of rigid plane surface in uniform rotation parallel to itself

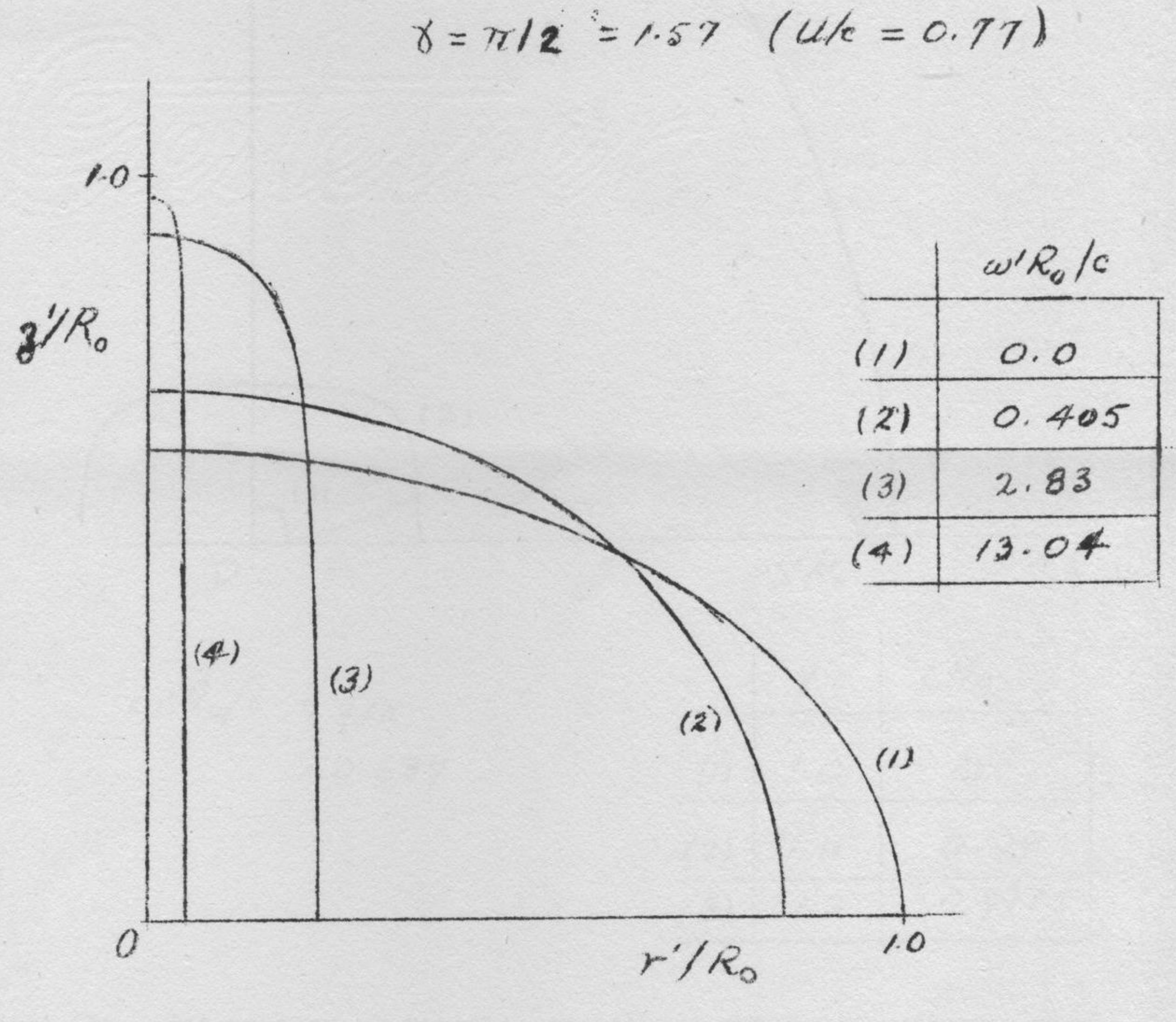
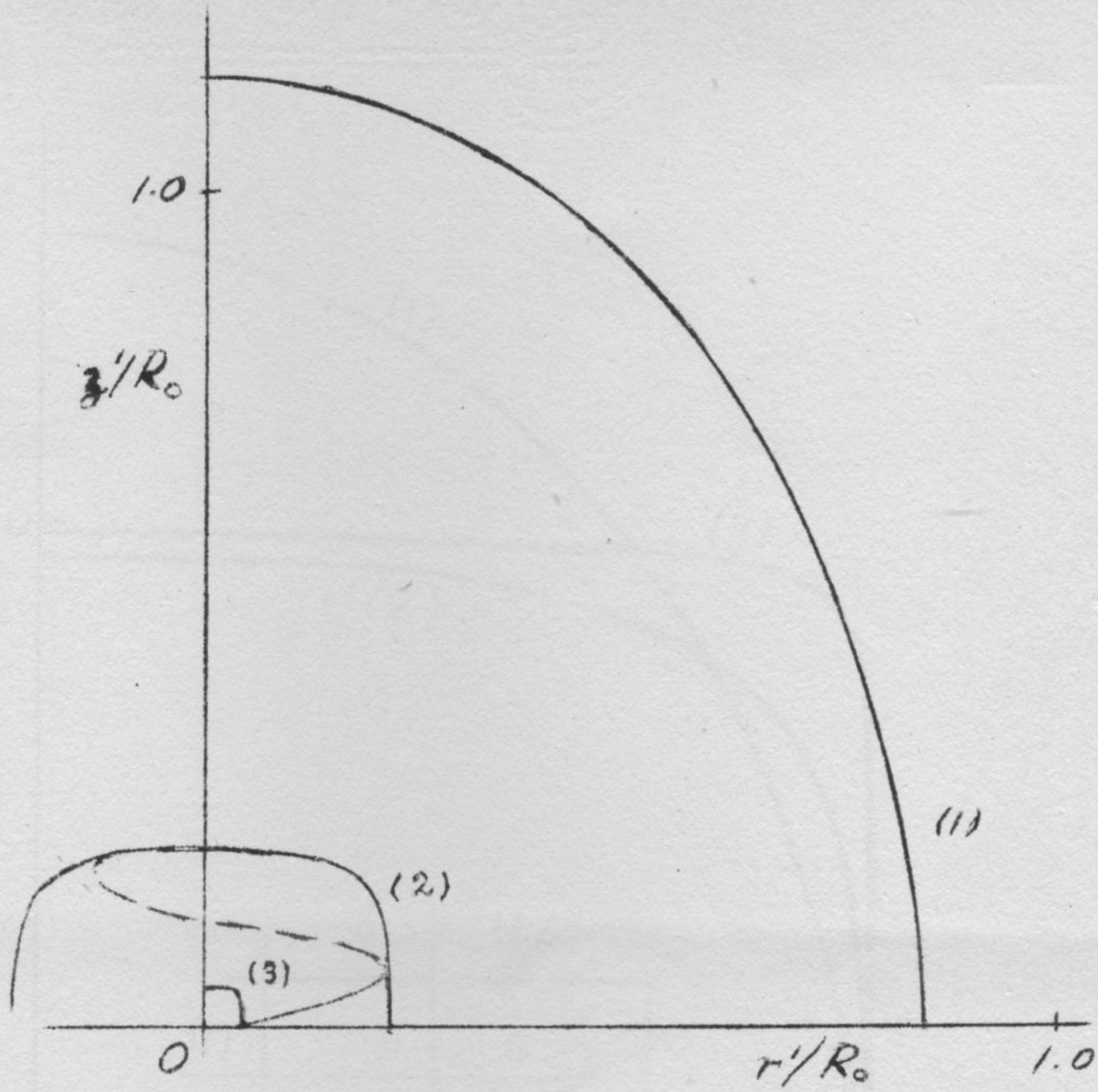


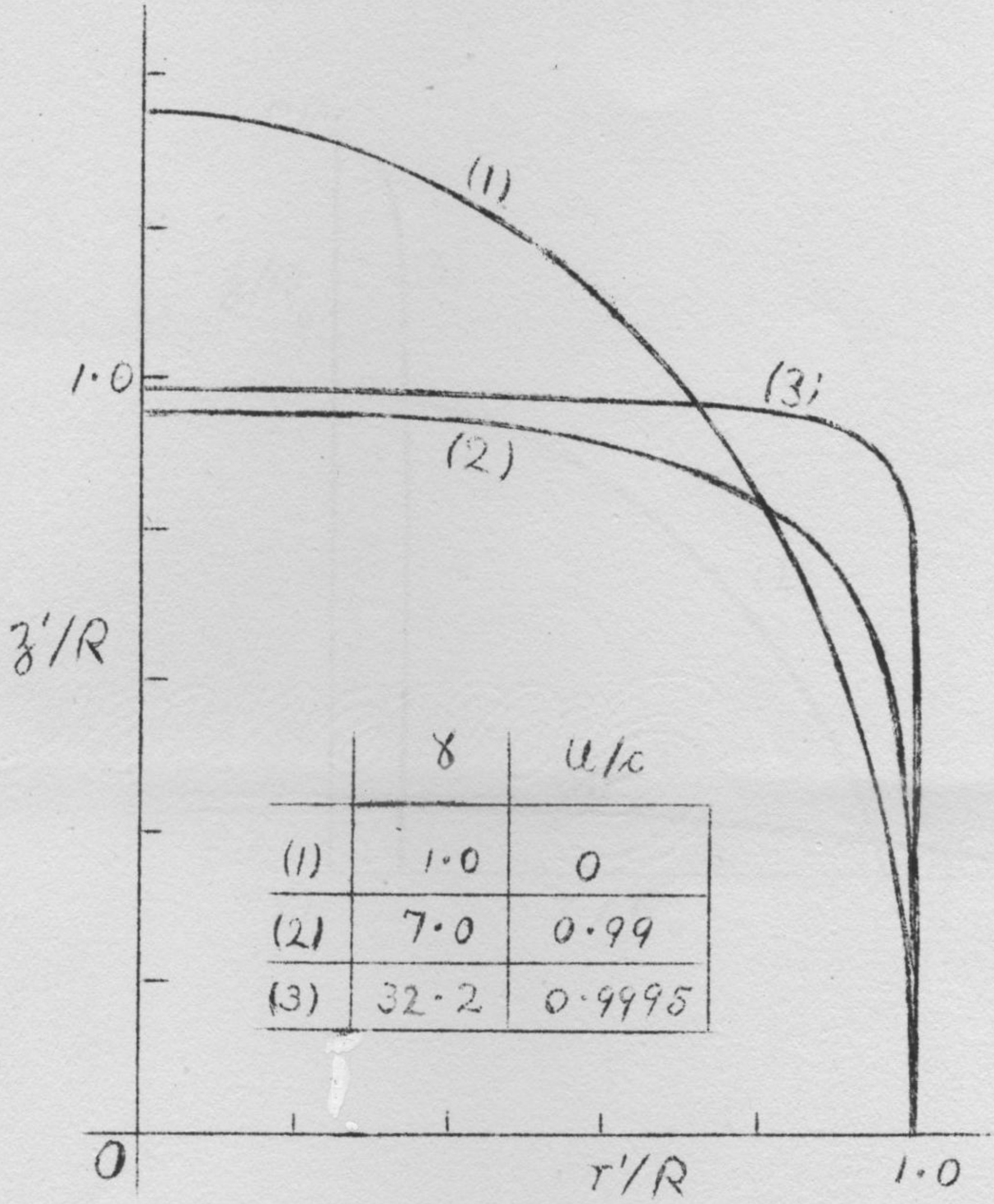
Fig. 6C: Deformation of meridian section of sphere in uniform screw motion (γ fixed).



$$\omega' R_0/c = 2/\pi = 0.637$$

	γ	U/c
(1)	1.0	0
(2)	7.0	0.99
(3)	32.2	0.9995

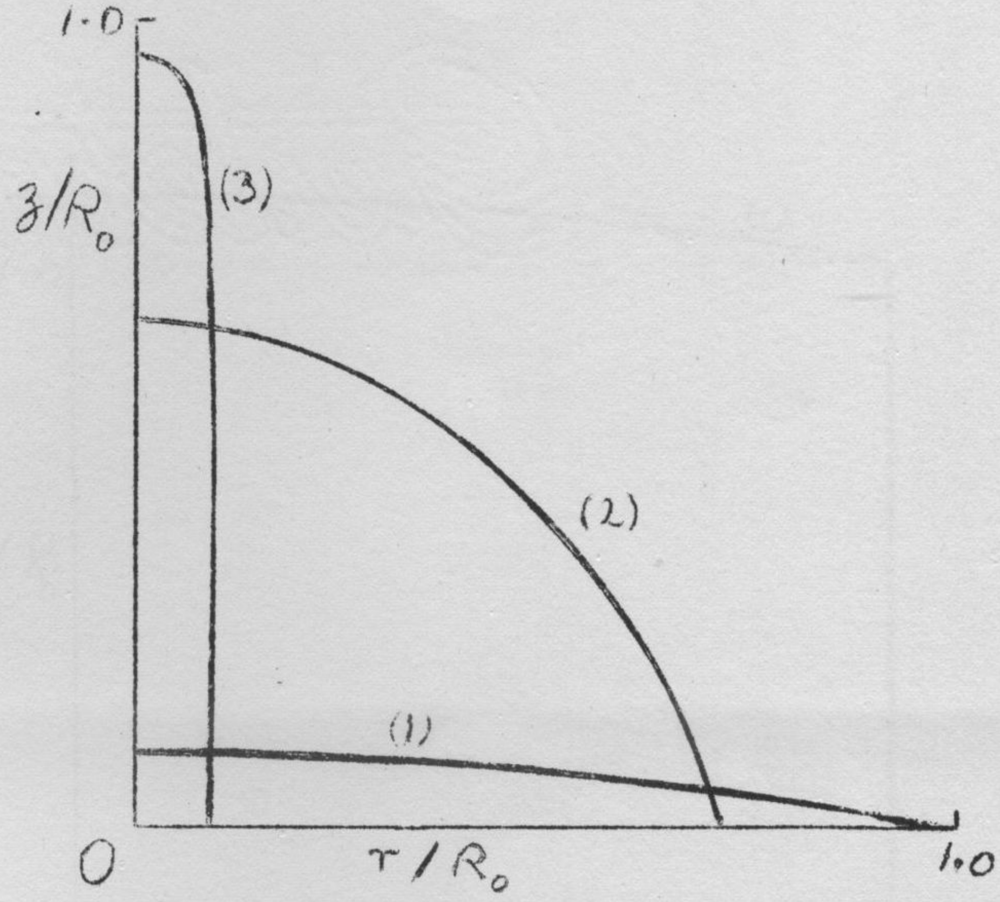
Fig. 6D: Deformation of meridian section of sphere in uniform screw motion (ω' fixed). (The simultaneous twisting of the sphere about the axis is indicated for $\gamma = 7\cdot 0$.)



	γ	u/c
(1)	1·0	0
(2)	7·0	0·99
(3)	32·2	0·9995

$$\omega' R_0 / c = 2/\pi = 0{\cdot}637$$

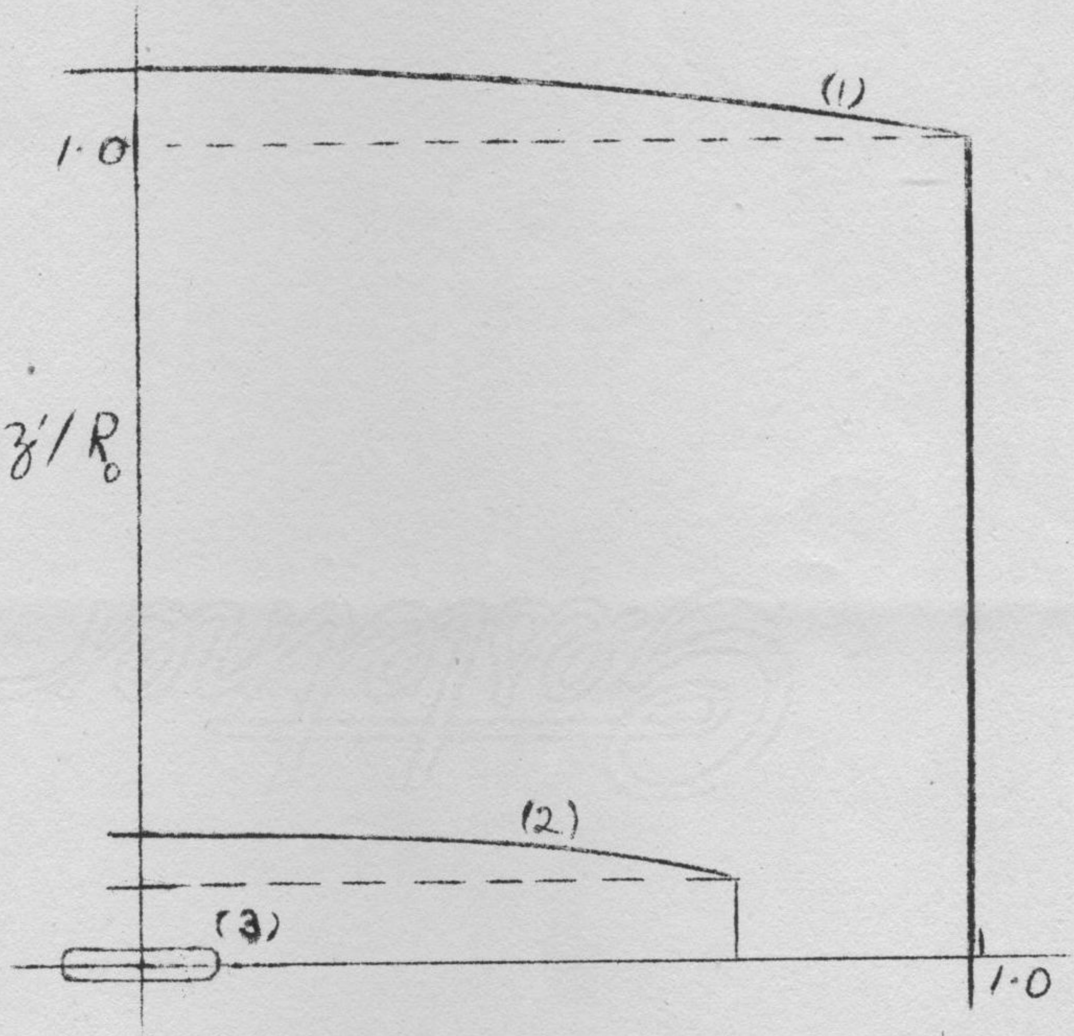
Fig. 6E: Meridian section of sphere in uniform screw motion enlarged by a factor $1/R \simeq \gamma\omega'/c$ $(R = r'_{max})$



(1) 0·1
(2) 1·0
(3) 10·0

$\left.\begin{array}{ll} (1) & 0\cdot1 \\ (2) & 1\cdot0 \\ (3) & 10\cdot0 \end{array}\right\} \omega' R_0/c$

Fig. 7A: Deformation of meridian section of flat end (cap) of circular cylinder in uniform rotation



$\omega' R_0 / c = 0{\cdot}1$

	γ	u/c
(1)	1	0
(2)	10	·995
(3)	100	·99995

Fig. 7B: Deformation of meridian section of circular cylinder in uniform screw motion (ω' fixed)